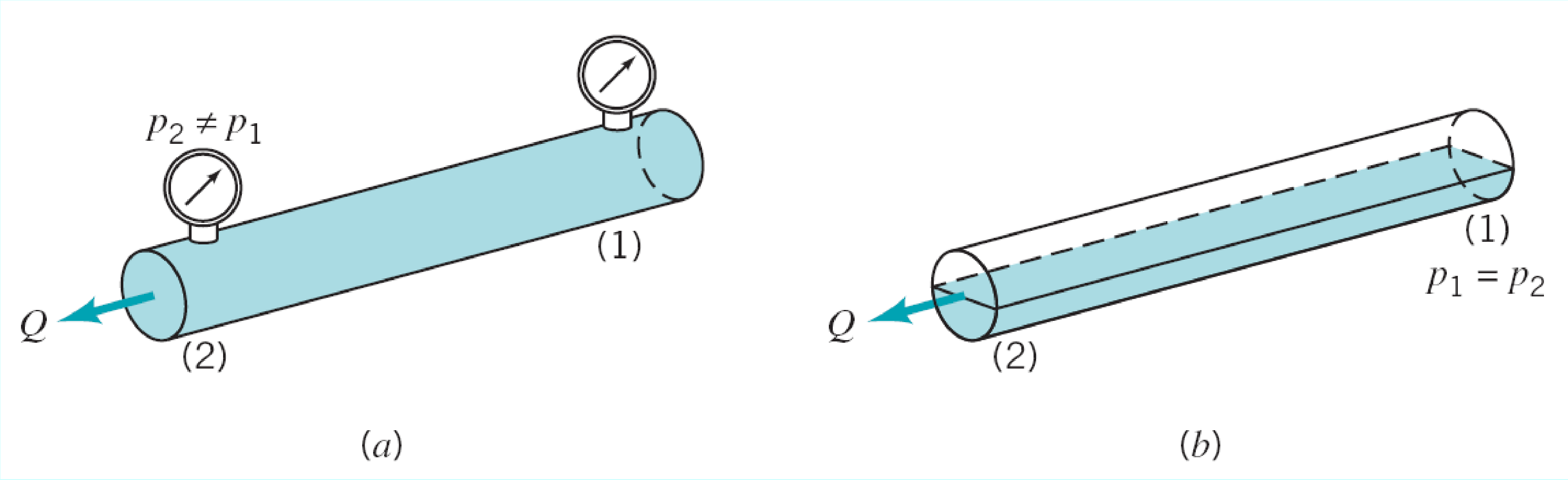
Chapter 8

Viscous Flow in Pipes

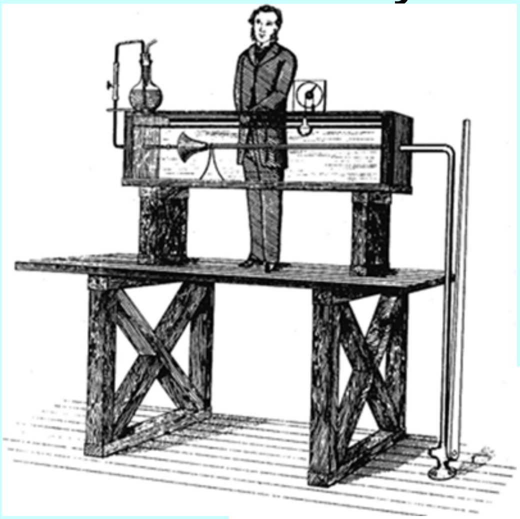
8.1 General Characteristics of Pipe Flow

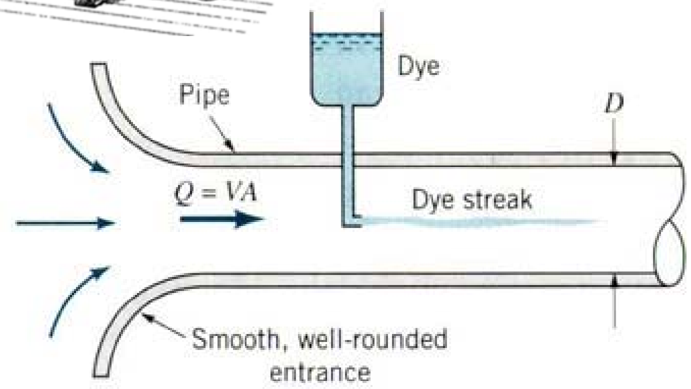
- We will consider round pipe flow, which is completely filled with the fluid.

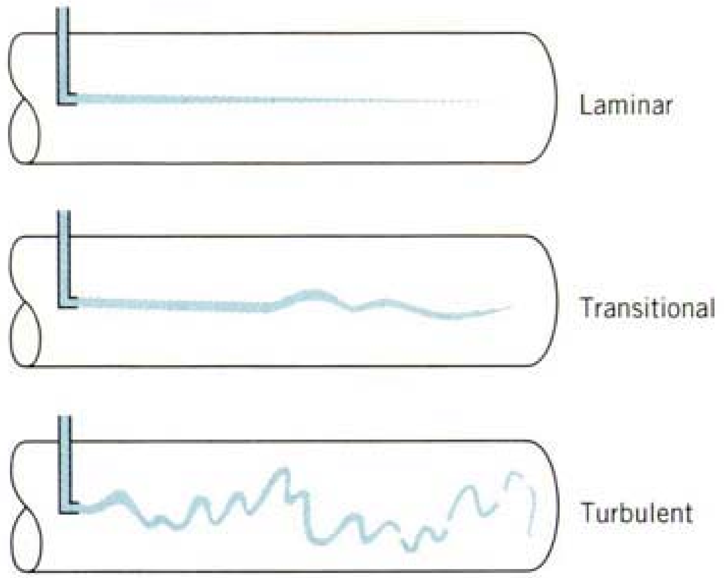


8.1.1 Laminar or Turbulent Flow

- The flow of a fluid in a pipe may be laminar flow or it may be turbulent flow.
- Osborne Reynolds experiment --typical dye streaks



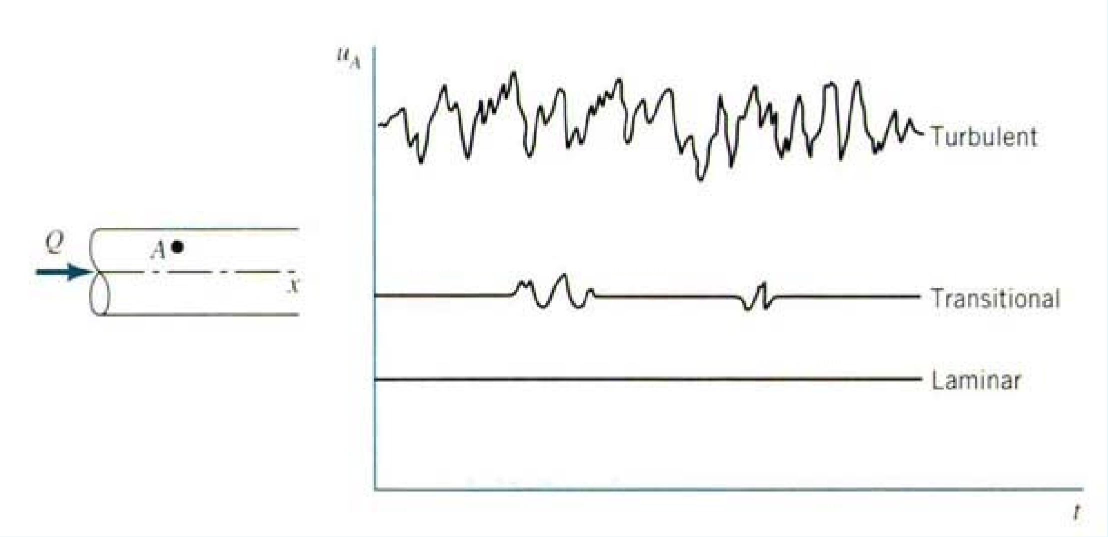




V8.2 Laminar/turbulent pipe flow

Instantaneous velocity fluctuation

- Time dependence of fluid velocity at a point.



Turbulent
Re>4000
Transitional
2100<Re<4000
Laminar
Re<2100

V8.3 Intermittent turbulent burst in pipe flow

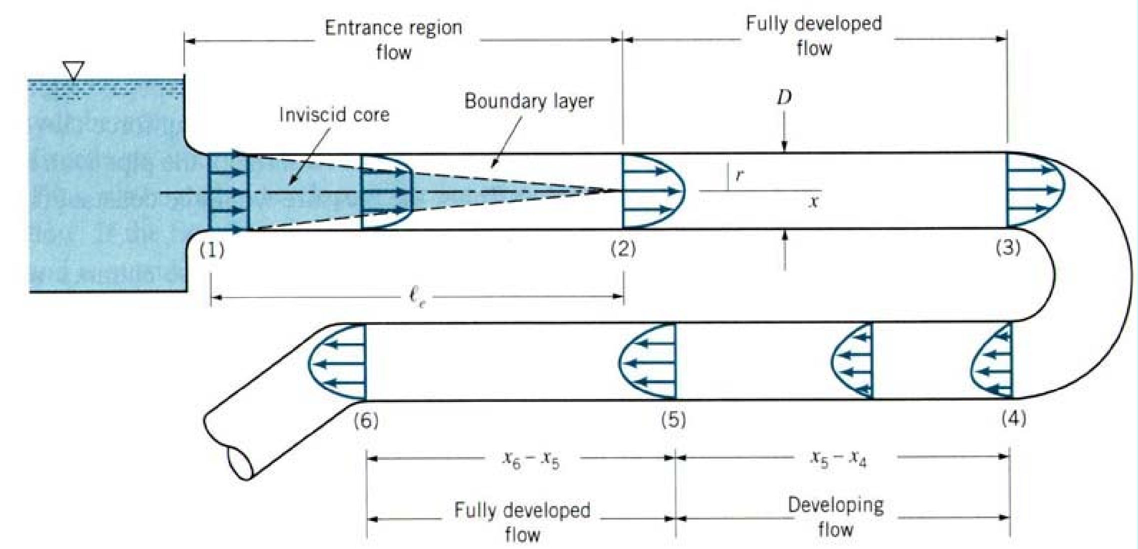
- For pipe flow the most important dimensionless parameter is the Reynolds number, Re, the rate of the inertia to viscous effect in the flow.

$$\mathrm{Re} = \frac{\rho V D}{\mu}$$

EX. 8.1 Laminar or turbulent flows?

8.1.2 Entrance Region and Fully Developed Flow

- Consider a pipe flow with uniform inlet velocity



A boundary layer in which viscous effect are important is produced along the pipe wall such that the initial velocity profile changes with distance along the pipe.

Entrance length

- Typical entrance lengths are given by

$$\frac{\ell_e}{D} = 0.06\,\text{Re} \qquad \text{for laminar flow}$$

$$\frac{\ell_e}{D} = 4.4\left(\text{Re}\right)^{1/6} \qquad \text{for turbulent flow}$$

e.g. $\text{Re} = 1.0 \qquad \ell_e = 0.6D$

$\text{Re} = 2000 \qquad \ell_e = 120D$

For practical engineering problems

$$10^4 < \text{Re} < 10^5 \quad \text{so that} \quad 20D < \ell_e < 30D$$

- Calculation of the velocity profile and pressure distribution within the entrance region is quite complex.

8.1.3 Pressure and Shear Stress

- Fully developed steady flow in a constant diameter pipe may be driven by gravity and/or pressure force.
- For horizontal pipe flow, gravity has no effect except for a hydrostatic pressure variation across the pipe, that is usually negligible.
- Viscous effects provide the restraining force that exactly balances the pressure force, allowing the fluid to flow through the pipe with no acceleration.
- In non-fully developed flow, the fluid accelerates or decelerates as it flows, therefore there is a balance between pressure, viscous and inertia (acceleration) flow.

Integral Analysis for Increased Pressure Drop in the Entrance Region

(From S.R. Turns, Thermal-Fluid Sciences, Cambridge Univ. Press, 2006)

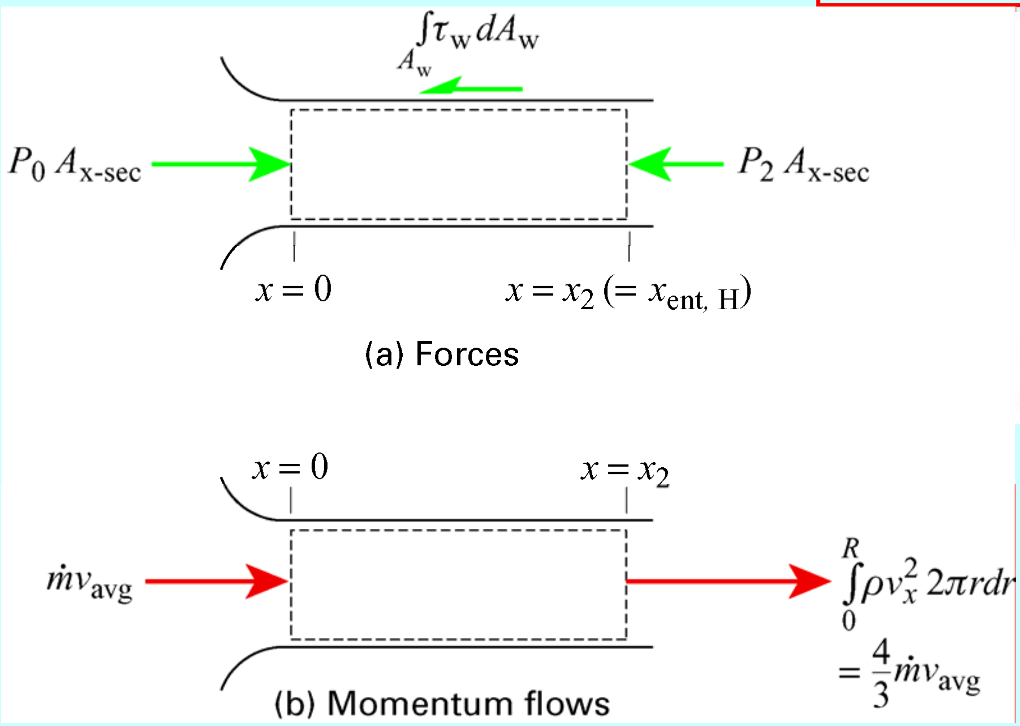
The increased pressure drop in the entrance region result from two reasons: 1. flow acceleration, 2. increased wall shear stress.

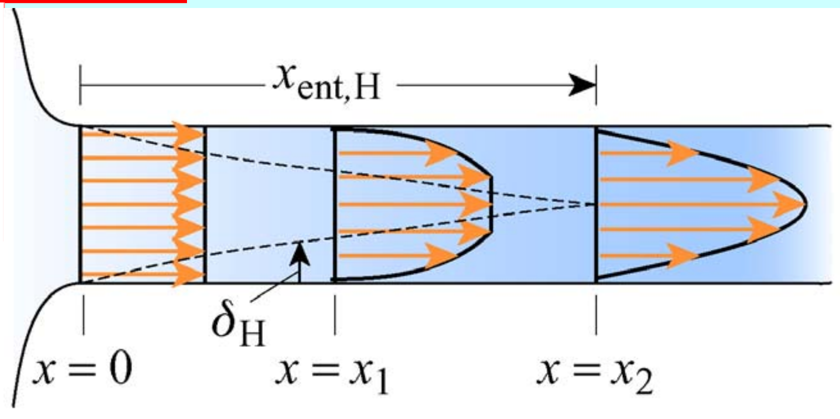
1. Flow acceleration

Integral momentum balance for the control volume below leads to

$$P_0 A_{x\text{-sec}} - P_2 A_{x\text{-sec}} - \int_{A_w} \tau_w \, dA_w = \boxed{\frac{4}{3}\dot{m}v_{avg} - \dot{m}v_{avg}}$$

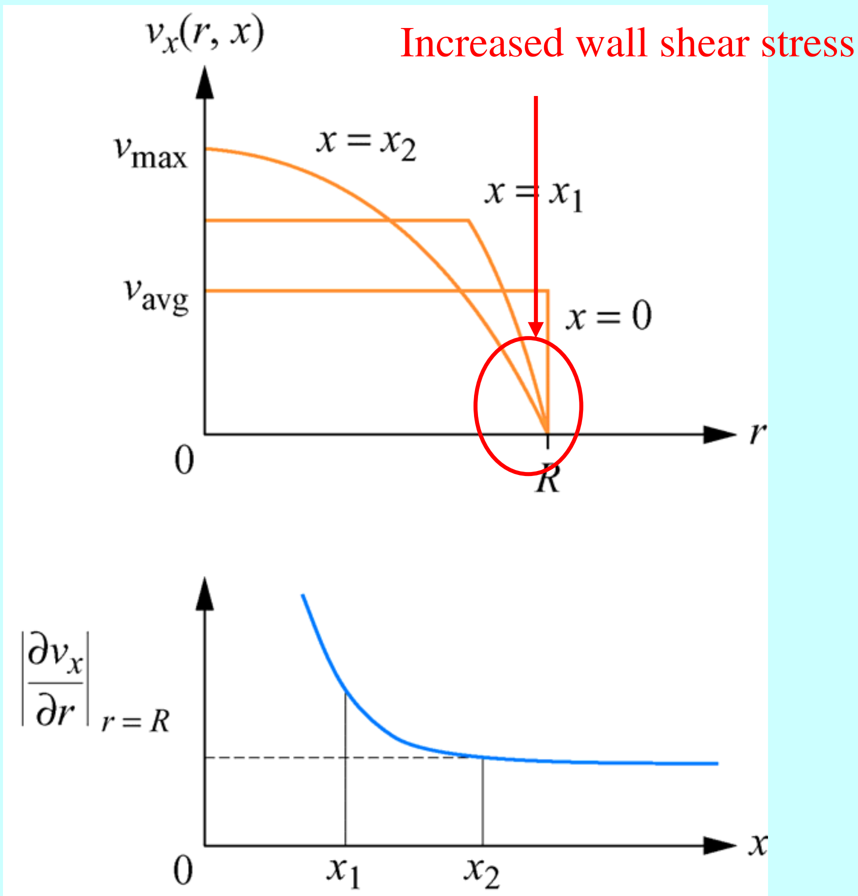
Increased momentum flow or flow acceleration





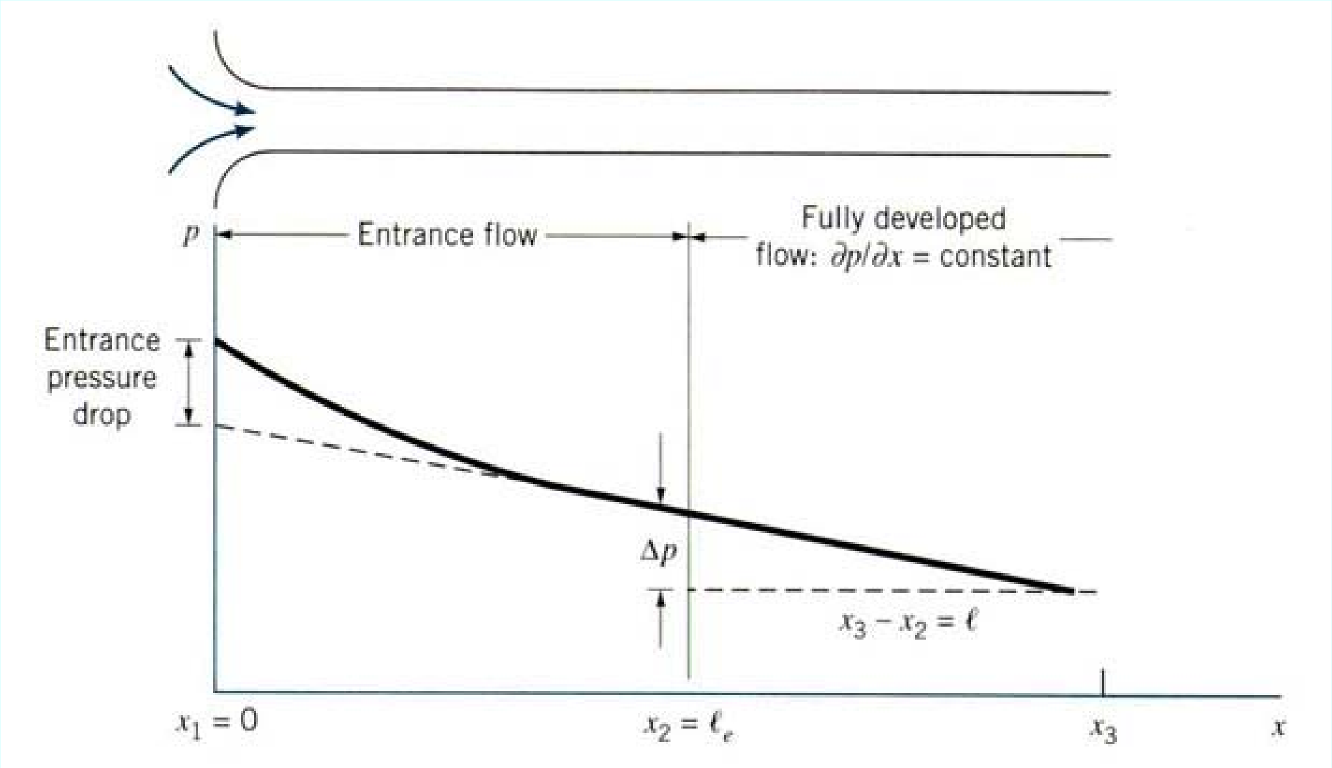
2. Increased Wall Shear Stress:

$$\left[\int_{A_\mathrm{w}} \tau_\mathrm{w}\, dA_\mathrm{w}\right]_{\substack{\text{developing}\\\text{region}}} > \left[\int_{A_\mathrm{w}} \tau_\mathrm{w}\, dA_\mathrm{w}\right]_{\substack{\text{fully}\\\text{developed}}}$$



Pressure distribution

- Pressure distribution drag a horizontal pipe



$$\left|\frac{\partial p}{\partial x}\right|_{\substack{\text{entrance}\\\text{region}}} > \left|\frac{\partial p}{\partial x}\right|_{\text{fully developed region}} = \text{constant}$$

Nature of flow

- The nature of the pipe flow is strongly dependent on whether the flow is laminar or turbulent, which is a direct consequence of the differences in the nature of the shear stress.
- Flow rate as a function of pressure drop in a given horizontal pipe.

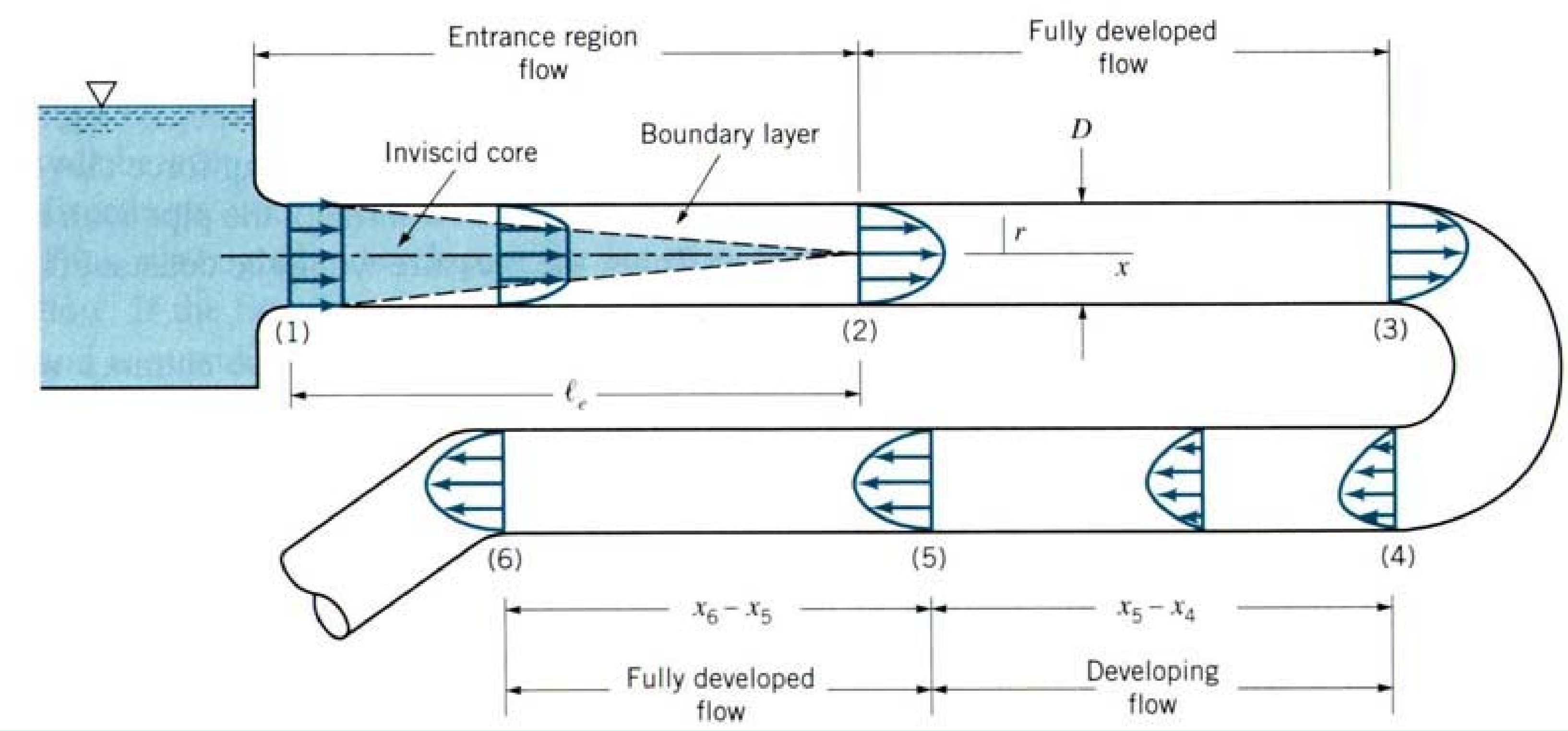
$$V = \frac{\Delta p \cdot D}{32 \cdot \mu \cdot \ell}$$

$$Q = \frac{\pi D^4 \Delta p}{128 \mu \ell} \quad - \quad \text{Hagen-Poiseuille flow}$$

$$\text{Laminar flow Re} < 2100$$

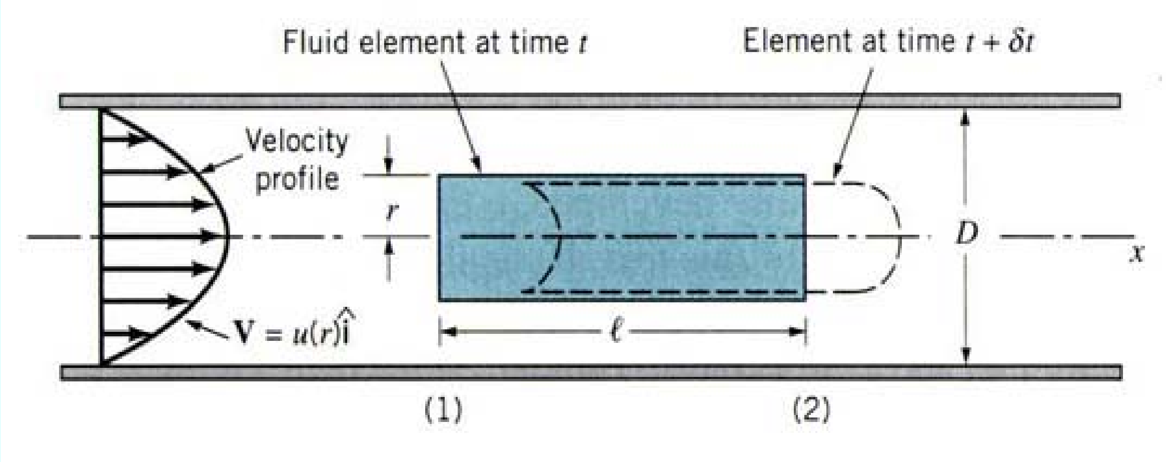
8.2 Fully Developed Laminar Flow

- Consider a fully developed flow in long straight, constant diameter sections of a pipe.
- There are numerous ways to derive important results pertaining to fully developed laminar flow.

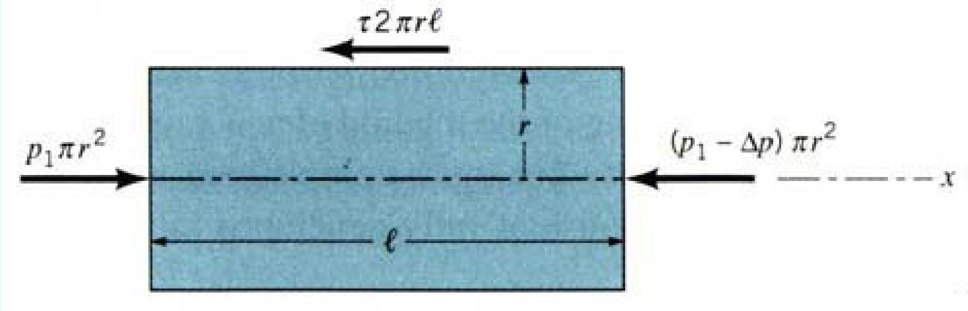


8.2.1 From *F=ma* Applied Directly to a Fluid Element

- Consider a element within a pipe



- Free body diagram of a cylinder of fluid



- The force balance becomes, (*a*=0, the fluid is not accelerating)

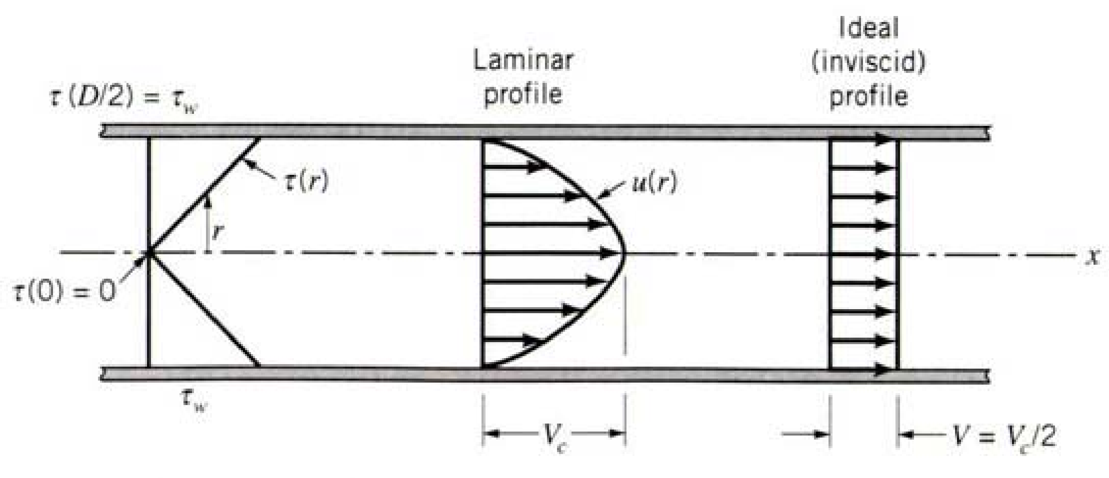
$$(p_1)\pi r^2 - (p_1 - \Delta p)\pi r^2 - (\tau)2\pi r\ell = 0 \quad \rightarrow \quad \frac{\Delta p}{\ell} = \frac{2\tau}{r}$$

- Since neither Δp nor ℓ are functions of the radial coordinate, r, it follows that $2\tau / r$ must also be independent of r.

i.e., $\tau = Cr$ where C is a constant

$$C = \frac{2\tau_w}{D} \quad \text{where } \tau_w \text{ is the wall shear stress}$$

$$\rightarrow \tau = \frac{2\tau_w r}{D}$$



- Thus the pressure drop and wall shear stress are related by, $$\Delta p = \frac{4\ell\tau_w}{D}$$

- A small shear stress can produce a large pressure difference if the pipe is relatively long $\ell / D >> 1$

- The above analysis is valid for both laminar and turbulent flow.
- To proceed further we must determine the relationship between shear stress and velocity.
- For laminar Newtonian flow,

$$\tau = \mu \frac{\partial u}{\partial y}$$

For pipe flow $\tau = -\mu \frac{du}{dr} \quad (\tau > 0)$

Then $\frac{\Delta p}{\ell} = \frac{2\tau}{r}, \quad \frac{du}{dr} = -\left(\frac{\Delta p}{2\mu\ell}\right) r$

$$\int du = -\frac{\Delta p}{2\mu\ell} \int r dr$$

$$u = -\left(\frac{\Delta p}{2\mu\ell}\right) r^2 + C_1, \qquad \text{where } C_1 \text{ is a constant}$$

$$\text{B.C.:} \quad u = 0., \quad r = \frac{D}{2}, \quad \text{so that} \quad C_1 = \frac{\Delta p D^2}{16\mu\ell}$$

- Hence the velocity profile can be written as,

$$u(r) = \frac{\Delta p D^2}{16\mu\ell}\left[1-\left(\frac{2r}{D}\right)^2\right] = V_c\left[1-\left(\frac{2r}{D}\right)^2\right]$$

where $V_c = \dfrac{\Delta p D^2}{16\mu\ell}$ is the centerline velocity

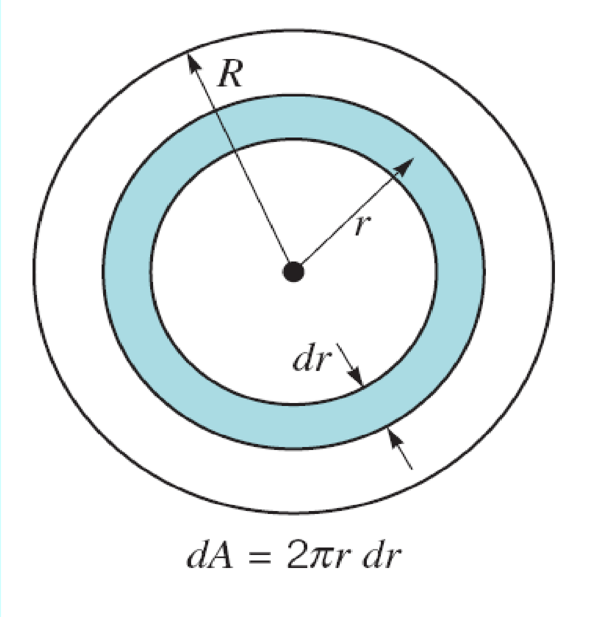
In terms of wall shear stress,

$$u(r) = \frac{\tau_w D}{4\mu}\left[1-\left(\frac{r}{R}\right)^2\right]$$ where $R = \dfrac{D}{2}$ is the pipe radius.

- Volume flow rate

$$Q = \int u\,dA = \int_{r=0}^{r=R} u(r)\,2\pi r\,dr = 2\pi V_c \int_0^R \left[1-\left(\frac{r}{R}\right)^2\right] r\,dr$$

$$= \frac{\pi R^2 V_c}{2}$$



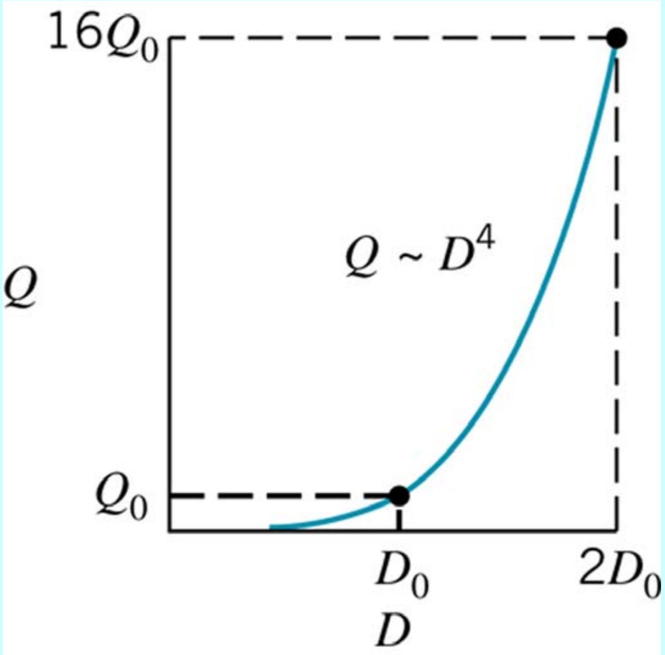
- Average velocity

$$V = \frac{\pi R^2 V_c}{2\pi R^2} = \frac{V_c}{2} = \frac{\Delta p \cdot D^2}{32 \cdot \mu \cdot \ell}$$

$$Q = \frac{\pi D^4 \Delta p}{128 \mu \ell} \quad - \text{ Hagen-Poiseuille flow}$$

Laminar flow Re<2100

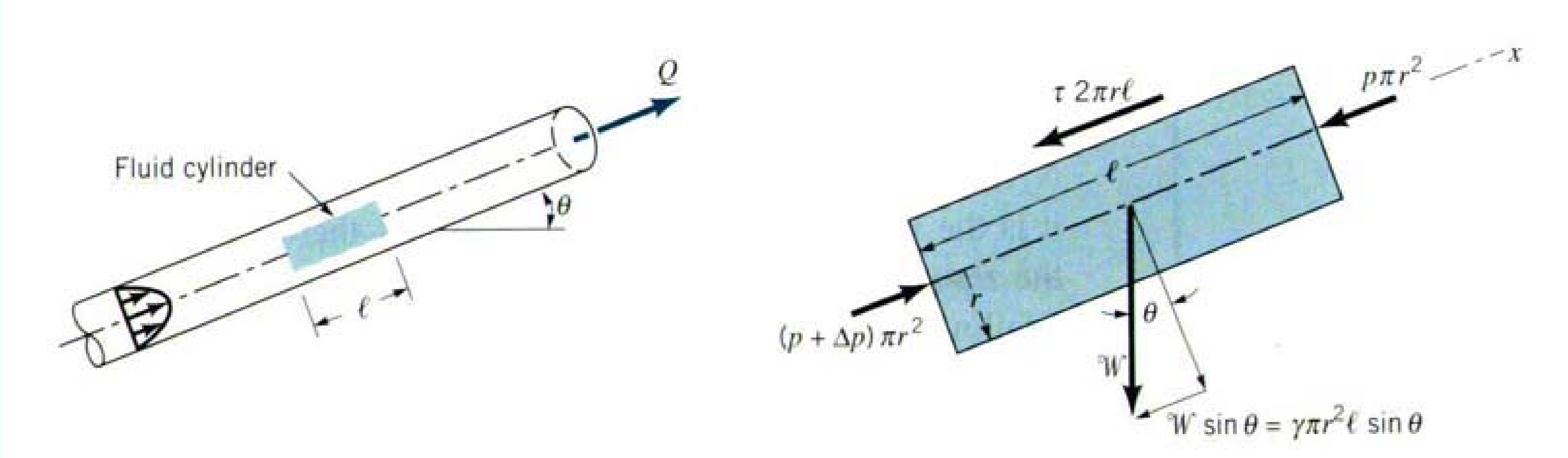
A 2% error in pipe diameter gives rise to 8% error in flow rate.



$$Q \sim D^4, \quad \delta Q \sim 4D^3 \delta D$$

$$\frac{\delta Q}{Q} = \frac{4\delta D}{D}$$

- For nonhorizontal pipe



Thus instead of $$\frac{\Delta p}{\ell} = \frac{2\pi}{r}$$

$$\frac{\Delta p - \gamma \ell \sin\theta}{\ell} = \frac{2\tau}{r}$$

and

$$V = \frac{\left(\Delta p - \gamma \ell \sin\theta\right) D^2}{32\mu\ell}, \quad Q = \frac{\pi\left(\Delta p - \gamma \ell \sin\theta\right) D^4}{128\mu\ell}$$

If $V = 0, \quad \Delta p = \gamma \ell \sin\theta = \gamma \Delta z$

8.2.2 From the Navier-Stokes Equation

- Basic equations

$$\nabla \cdot \vec{V} = 0,$$

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} = -\frac{\nabla p}{\rho} + \vec{g} + \nu \nabla^2 \vec{V}$$

- For steady fully developed flow in a pipe

$$\nabla \cdot \vec{V} = 0,$$

$$\nabla p + \rho g \vec{k} = \mu \nabla^2 \vec{V}$$

- In terms of polar coordinate

$$\frac{\partial p}{\partial x} + \rho g \sin\theta = \mu \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r} \right)$$

$$f(x) = g(r) = \text{constant},$$

8.2.3 From Dimensional Analysis

- Consider pressure drop in the horizontal pipe

$$\Delta p = F\left(V, \ell, D, \mu\right)$$

$$k - r = 5 - 3 = 2 \qquad \text{dimensionless groups}$$

$$\frac{D \cdot \Delta p}{\mu \cdot V} = \phi\left(\frac{\ell}{D}\right)$$

- If we assume that the pressure drop is directly proportional to the pipe length.

$$\frac{D \cdot \Delta p}{\mu \cdot V} = \frac{C\ell}{D} \quad , \quad \frac{\Delta p}{\ell} = \frac{C\mu V}{D^2} \Rightarrow V = \frac{\Delta p}{\ell}\frac{D^2}{C\mu}$$

$$\text{or} \quad Q = AV = \frac{\left(\pi/4C\right)\Delta p \cdot D^4}{\mu\ell} = \frac{\pi D^4 \cdot \Delta p}{128\mu\ell}$$

The value of *C* must be determined by theory or experiment.
C = 32 for round pipe.

- It is advantageous to describe in terms if dimensionless quantities.

$$\Delta p = \frac{32\mu\ell V}{D^2}$$

$$\frac{\Delta p}{\frac{1}{2}\rho V^2} = \frac{32\mu\ell V / D^2}{\frac{1}{2}\rho V^2} = 64\frac{\mu}{\rho V D}\frac{\ell}{D} = \frac{64}{\mathrm{Re}}\left(\frac{\ell}{D}\right)$$

$$\text{i.e.,} \quad \Delta p = f \cdot \frac{\ell}{D} \cdot \frac{\rho V^2}{2}$$

where $f = \Delta p \left(D/\ell \right) \big/ \left(\rho V^2 / 2 \right)$ friction factor or Darcy friction factor

For laminar fully developed pipe flow $f = \dfrac{64}{\mathrm{Re}}$ (8.19)

- Alternatively, since $\Delta p = \dfrac{4 \cdot \ell \tau_w}{D}$

$$f = \frac{\Delta p \cdot \left(D/\ell \right)}{\frac{1}{2}\rho V^2} = \frac{4 \cdot \ell \tau_w}{D}\frac{D}{\ell}\frac{1}{\frac{1}{2}\rho V^2} = \frac{8\tau_w}{\rho V^2} \qquad (8.20)$$

8.2.4 Energy Consideration

- Consider the energy equation for incompressible, steady flow between flow locations.

$$\frac{p_1}{\gamma}+\alpha_1\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\alpha_2\frac{V_2^2}{2g}+z_2+h_2$$

- For fully developed flow $\alpha_1=\alpha_2\ ,\quad V_1=V_2$

$$\frac{p_1}{\gamma}+z_1-\left(\frac{p_2}{\gamma}+z_2\right)=h_L$$

$$\frac{\Delta p}{\ell}=\frac{2\tau}{r}\qquad \therefore h_L=\frac{2\tau\ell}{\gamma r}\quad ,\ h_L=\frac{4\ell\,\tau_w}{\gamma D}$$

$$\left(\begin{array}{l}\dfrac{\Delta p-\gamma\ell\sin\theta}{\ell}=\dfrac{2\tau}{r}\\ \dfrac{\Delta p}{\gamma}-\dfrac{\ell\sin\theta}{\ell}=\dfrac{2\tau}{r\gamma}\end{array}\right)$$

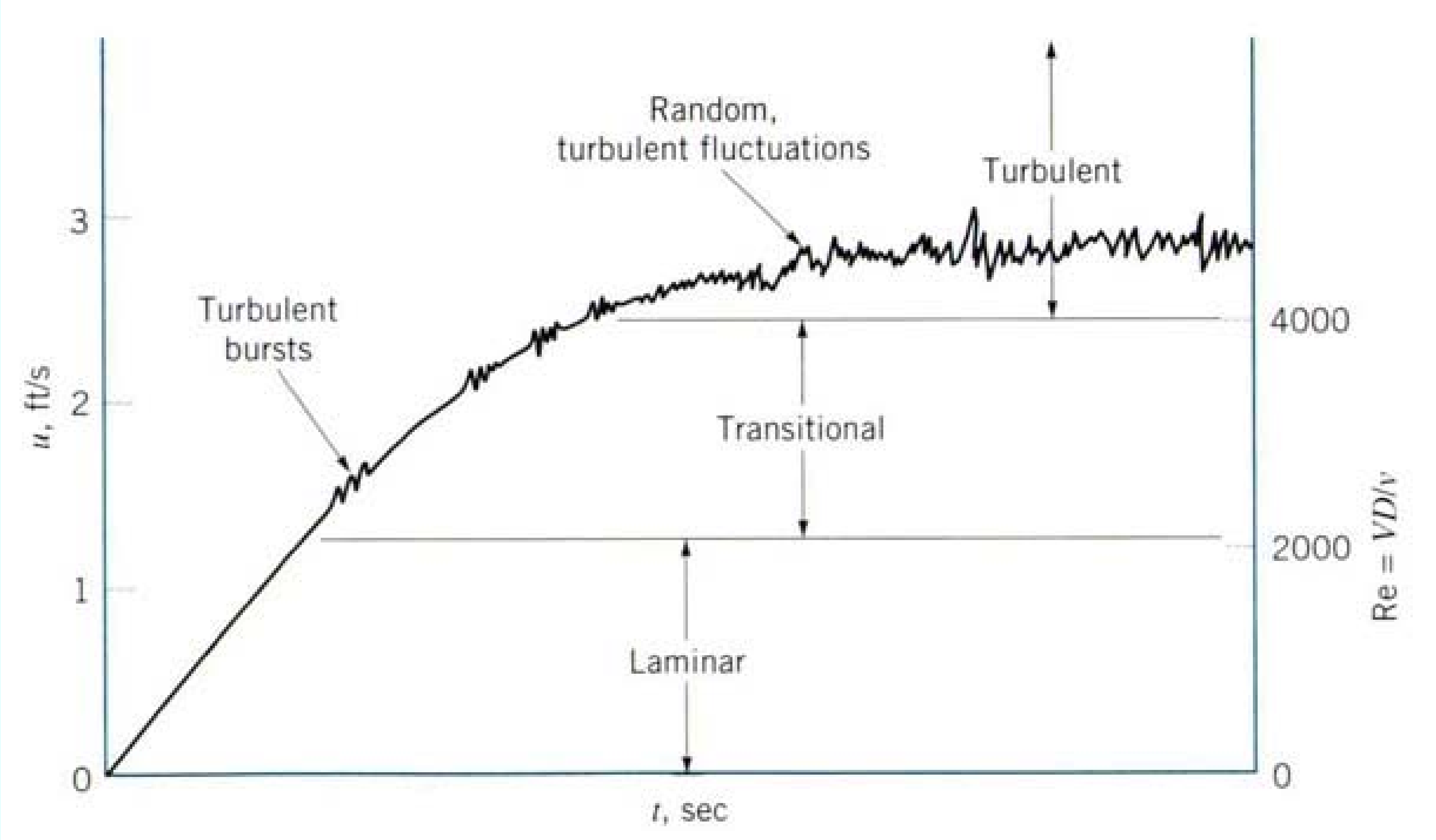
(Recall $p_1=p_2+\Delta p\ ,\ z_2-z_1=\ell\sin\theta$)

Which indicates that it is the shear stress at the wall (which is directly related to the viscosity and the shear stress throughout the fluid) that is responsible for the head loss.

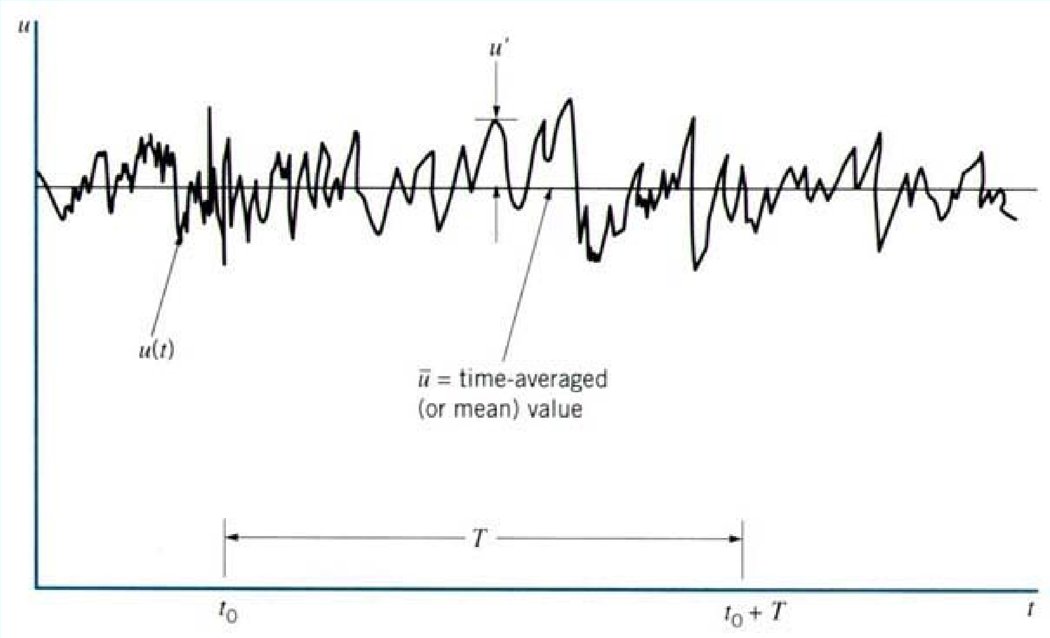
EX. 8.3 Laminar pipe flow properties

8.3 Fully developed turbulent flow

- Turbulent pipe flow is actually more likely to occur than laminar flow in practical situation.
- Consider flow in a pipe



- Velocity variation at a point.



- Turbulent processes and heat and mass transfer processes are considerably enhanced in turbulent flow compared to laminar flow.
  Mix a cup of coffee (turbulent flow)
  Mix two colors of a viscous paint (laminar flow)

V8.4 Stirring color into paint
V8.5 Laminar and turbulent mixing
V8.7 Turbulence in a bowl

8.3.2 Turbulent shear stress

- Turbulent flow is random and chaotic and can only be characterized using stately terms.

$$u = u\left(x, y, z, t\right)$$

- Mean part

time average $$\overline{u} = \frac{1}{T}\int_{t_0}^{t_0+T} u\left(x, y, z, t\right) dt$$

where *T* is longer than the period of the longest fluctuations, but is shorter than any unsteadiness of the average velocity.

Turbulence intensity

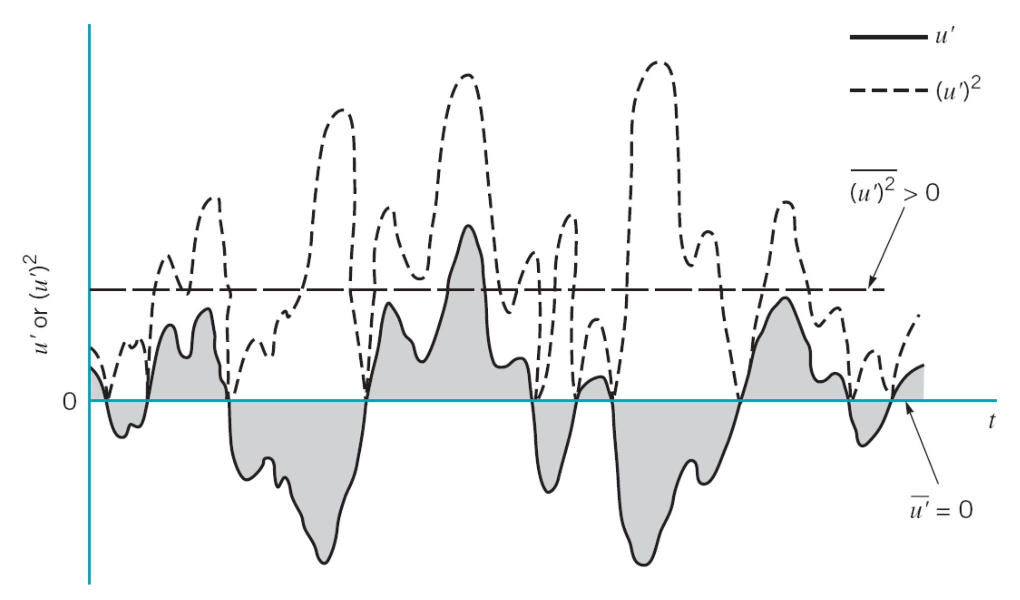
- Fluctuating part

$$u = \overline{u} + u' \quad \text{or} \quad u' = u - \overline{u}$$

$$\overline{u'} = \frac{1}{T}\int_{t_0}^{t_0+T} (u - \overline{u})\, dt = \frac{1}{T}\left[\int_{t_0}^{t_0+T} u\, dt - \overline{u}\int_{t_0}^{t_0+T} dt\right]$$

$$= \frac{1}{T}\left(T\overline{u} - T\overline{u}\right) = 0$$

$$\overline{u'^2} = \frac{1}{T}\int_{t_0}^{t_0+T} u'^2\, dt \; > 0$$



- Turbulence intensity

$$I = \frac{\sqrt{\overline{u'^2}}}{\overline{u}} = \frac{\left[\frac{1}{T}\int_{t_0}^{t_0+T} u'^2\, dt\right]^{1/2}}{\overline{u}}$$

Well designed wind tunnel: $I \approx 0.01$ (may be down to 0.0002)

Time-averaged Navier-Stokes Equations

Let $u = \bar{u} + u', \quad \upsilon = \bar{\upsilon} + \upsilon', \quad w = \bar{w} + w'$ and then take time average

$$x\text{-dir:}\quad \rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \overline{u'\frac{\partial u'}{\partial x}} + \bar{\upsilon}\frac{\partial \bar{u}}{\partial y} + \overline{\upsilon'\frac{\partial u'}{\partial y}} + \bar{w}\frac{\partial \bar{u}}{\partial z} + \overline{w'\frac{\partial u'}{\partial z}}\right) = -\frac{\partial \bar{p}}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right)$$

But $\overline{u'\frac{\partial u'}{\partial x}} = \frac{\partial \overline{u'^2}}{\partial x} - \overline{u'\frac{\partial u'}{\partial x}}; \quad \overline{\upsilon'\frac{\partial u'}{\partial y}} = \frac{\partial \overline{u'\upsilon'}}{\partial y} - \overline{u'\frac{\partial \upsilon'}{\partial y}}; \quad \overline{w'\frac{\partial u'}{\partial z}} = \frac{\partial \overline{u'w'}}{\partial z} - \overline{u'\frac{\partial w'}{\partial z}}$

and $\frac{\partial u'}{\partial x} + \frac{\partial \upsilon'}{\partial y} + \frac{\partial w'}{\partial z} = 0$ (from continuity equation)

$$\rightarrow\ \rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \frac{\partial \overline{u'^2}}{\partial x} + \bar{\upsilon}\frac{\partial \bar{u}}{\partial y} + \frac{\partial \overline{u'\upsilon'}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} + \frac{\partial \overline{u'w'}}{\partial z}\right) = -\frac{\partial \bar{p}}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right)$$

Terms with fluctuations moved and merged to the r.h.s.

Finally,

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{\upsilon}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z}\right) = -\frac{\partial \bar{p}}{\partial x} + \rho g_x + \frac{\partial}{\partial x}\left(\mu\frac{\partial \bar{u}}{\partial x} - \rho\overline{u'^2}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'\upsilon'}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial \bar{u}}{\partial z} - \rho\overline{u'w'}\right)$$

$$\rho\left(\frac{\partial \bar{\upsilon}}{\partial t} + \bar{u}\frac{\partial \bar{\upsilon}}{\partial x} + \bar{\upsilon}\frac{\partial \bar{\upsilon}}{\partial y} + \bar{w}\frac{\partial \bar{\upsilon}}{\partial z}\right) = -\frac{\partial \bar{p}}{\partial y} + \rho g_y + \frac{\partial}{\partial x}\left(\mu\frac{\partial \bar{\upsilon}}{\partial x} - \rho\overline{u'\upsilon'}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{\upsilon}}{\partial y} - \rho\overline{\upsilon'^2}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial \bar{\upsilon}}{\partial z} - \rho\overline{\upsilon'w'}\right)$$

$$\rho\left(\frac{\partial \bar{w}}{\partial t} + \bar{u}\frac{\partial \bar{w}}{\partial x} + \bar{\upsilon}\frac{\partial \bar{w}}{\partial y} + \bar{w}\frac{\partial \bar{w}}{\partial z}\right) = -\frac{\partial \bar{p}}{\partial z} + \rho g_z + \frac{\partial}{\partial x}\left(\mu\frac{\partial \bar{w}}{\partial x} - \rho\overline{u'w'}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{w}}{\partial y} - \rho\overline{\upsilon'w'}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial \bar{w}}{\partial z} - \rho\overline{w'^2}\right)$$

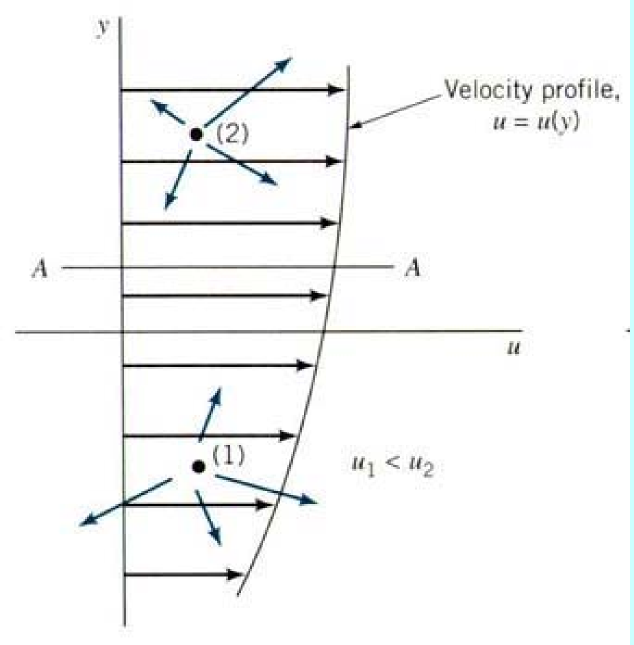
Origin of shear stress

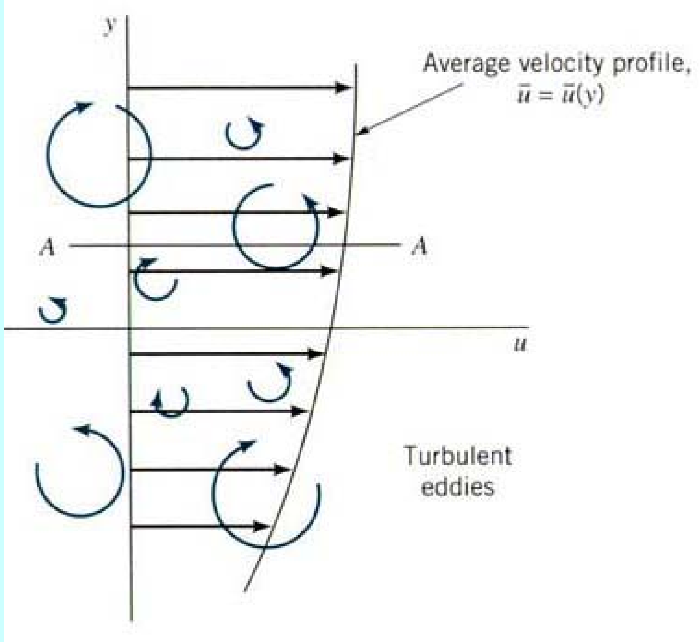
- The shear stress for turbulent flow can not be evaluated as for laminar flow, i.e.,

$$\tau_t \neq \mu \frac{\partial \overline{u}}{\partial y}$$

- Laminar flow : random motion of molecules.

Turbulent flow: random motion of 3D eddies.





The eddy structure greatly enhances and promotes mixing within the fluid.

Total stress

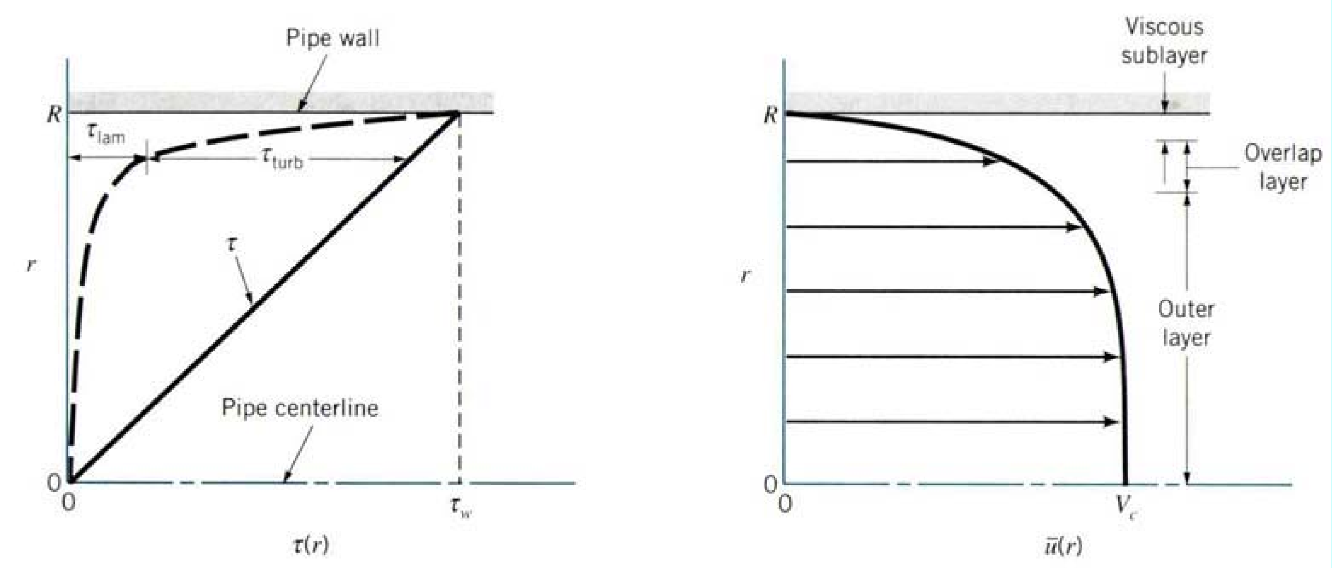
- Total stress

$$\tau = \mu \frac{d\bar{u}}{dy} - \rho \overline{u'\upsilon'} = \tau_{\text{lam}} + \tau_{\text{turb}}$$

if the flow is laminar, $u' = \upsilon' = 0$, $\overline{u'\upsilon'} = 0$

$-\rho\overline{u'\upsilon'}$ $(-\rho\overline{\upsilon'w'}, \ldots)$ **--Reynolds stresses**

- Structure of turbulent flow in a pipe



- Typically the value of τ_{turb} is 100 to 1000 times greater than τ_{lam} in the outer region, while the converse is true in the in the viscous sublayer.
- The viscous sublayer is usually a very thin layer.
  (cf. **Ex. 8.4**)
- An alternate way of expressing turbulent shear stress

$$\tau_{\text{turb}} = \eta \frac{d\bar{u}}{dy}$$

where η : **eddy viscosity** which is a function of both the fluid and flow conditions.

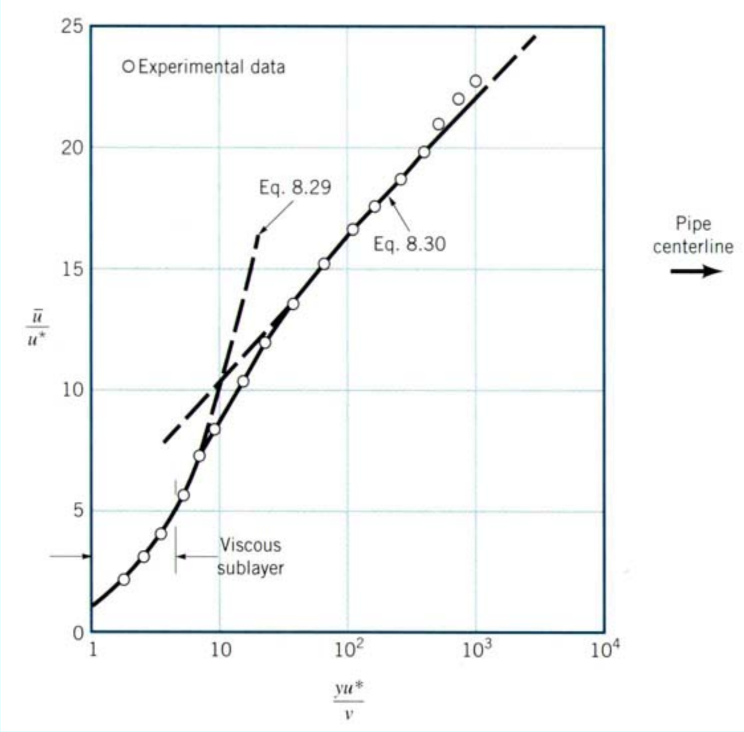
- **Prandtl mixing length**

$$\eta = \rho \ell_m^2 \left| \frac{d\bar{u}}{dy} \right|$$

$$\tau_{\text{turb}} = \rho \ell_m^2 \left(\frac{d\bar{u}}{dy} \right)^2 \qquad \ell_m = ?$$

8.3.3 Turbulent Velocity Profile

- Typical Structure of the turbulent velocity profile in a pipe



∗ law of the wall

$$\frac{\bar{u}}{u^*} = \frac{yu^*}{\nu}$$

$$\frac{\bar{u}}{u^*} = 2.5\ln\left(\frac{yu^*}{\nu}\right) + 5.0$$

where the coefficients are determined experimentally

$$u^* \text{ or } u_\tau = \sqrt{\frac{\tau_w}{\rho}}$$ friction velocity

viscous sublayer $0 \le yu^*/\nu \le 5$

In the central region, $\frac{V_c - \bar{u}}{u^*} = 2.5\ln\left(\frac{R}{y}\right)$ is often used

Appendix: Derivation of the law of the wall

For the sublayer, Prandtl suggested in 1930 that *u* be independent of sublayer thickness, and

$$u = f(\mu, \tau_w, \rho, y)$$

Using the pi method:

$$\Pi_1 = u\mu^a \tau_w^b \rho^c \rightarrow a = 0, b = -\frac{1}{2}, c = \frac{1}{2}$$

$$\Pi_2 = y\mu^d \tau_w^e \rho^f \rightarrow d = -1, e = \frac{1}{2}, c = \frac{1}{2}$$

$$\Rightarrow \Pi_1 = u / \sqrt{\tau_w / \rho} = u / u^*, \text{ where } \sqrt{\tau_w / \rho} \equiv u^*$$

$$\Pi_2 = \frac{y}{\nu}\sqrt{\tau_w / \rho} = \frac{yu^*}{\nu}$$

$$\rightarrow \frac{\bar{u}}{u^*} = f\left(\frac{yu^*}{\nu}\right)$$

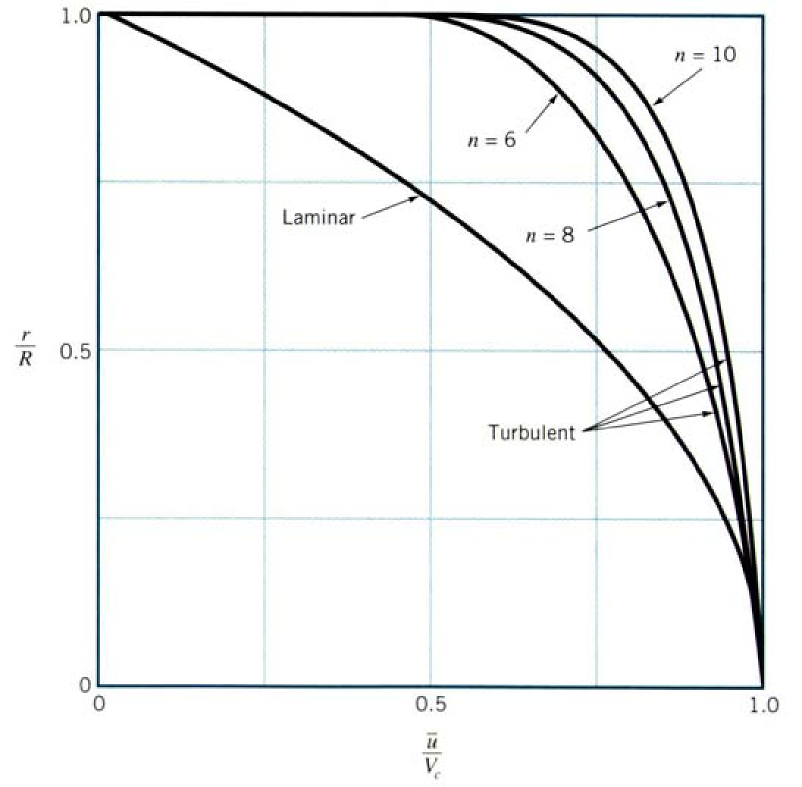
Experimentally, it is found that $\frac{\bar{u}}{u^*} = \frac{yu^*}{\nu}$.

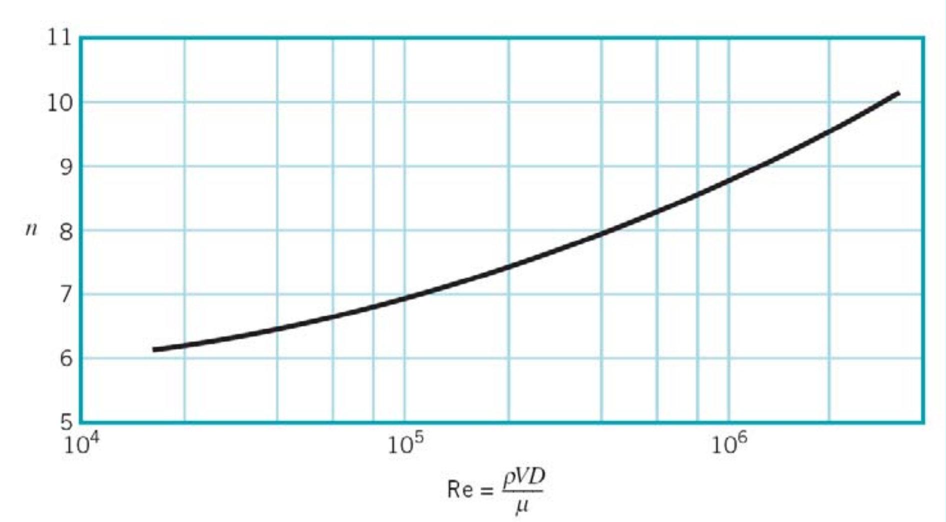
Turbulent Velocity Profile

- Empirical power–law velocity

$$\frac{\bar{u}}{V_c} = \left(1 - \frac{r}{R}\right)^{1/n}$$

the value of n is a function of the Reynolds number.





EX. 8.4 Turbulent pipe flow properties

V8.8 Laminar to turbulent flow from a pipe

V8.9 Laminar/turbulent velocity profiles

8.3.4 Turbulence Modeling

Time-averaging approach: need to solve the closure problem for the Reynolds stresses, such as $-\rho\overline{u'v'}$, etc.

Different methods for Reynolds stress closure:

• Algebraic (zero-equation) models—by modeling the eddy viscosity or mixing length

• One-equation models—solve one additional transport equation

• Two-equation models—solve two more additional transport equations, such as k-ε model, k-ω model

• Reynolds stress transport models—solve the transport equation for Reynolds stresses

Large eddy simulations (LES)—explicitly solve for the large eddies in a calculation and implicitly account for the small eddies by using a subgrid-scale model (SGS model).

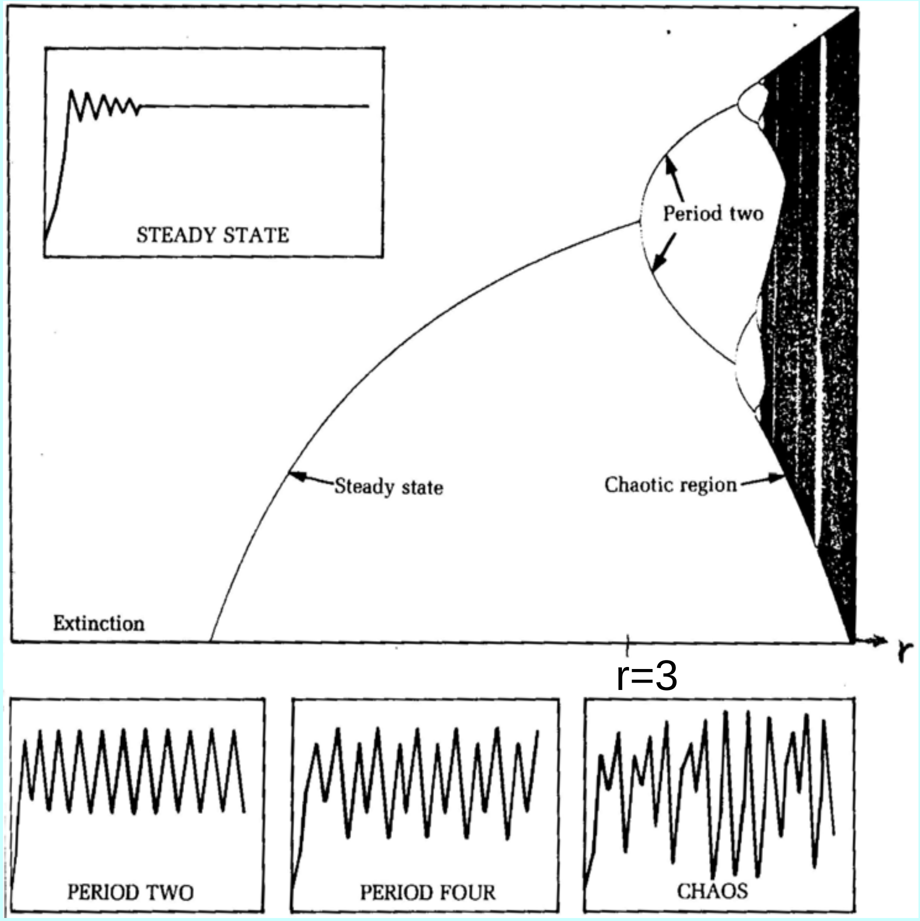
Direct numerical simulations (DNS)—directly solve the transient 3-D Navier-Stokes equations with very fine grids and time steps

8.3.5 Chaos and Turbulence

Chaos theory involves the behavior of nonlinear dynamical systems and their response to initial and boundary conditions. The flow of viscous fluid, governed by the complex nonlinear Navier-Stokes equations, is such a system.

The chaotic behavior associated with the logistic equation with increase of *r*:

$$X_{\text{next}} = rX(1 - X)$$



(from **Chaos**, by James Gleick)

The Butterfly Effect

The phrase refers to the idea that a butterfly's wings might create tiny changes in the atmosphere that may ultimately alter the path of a tornado or delay, accelerate or even prevent the occurrence of a tornado in a certain location. The flapping wing represents a small change in the initial condition of the system, which causes a chain of events leading to large-scale alterations of events. Had the butterfly not flapped its wings, the trajectory of the system might have been vastly different. Of course the butterfly cannot literally *cause* a tornado. The kinetic energy in a tornado is enormously larger than the energy in the turbulence of a butterfly. The kinetic energy of a tornado is ultimately provided by the sun and the butterfly can only influence certain details of weather events in a chaotic manner. (from Wikipedia)

About the difficulty to understand turbulence

On his death bed, **Heisenberg** is reported to have said, "When I meet God, I am going to ask him two questions: Why **relativity**? And why **turbulence**? I really believe he will have an answer for the first."

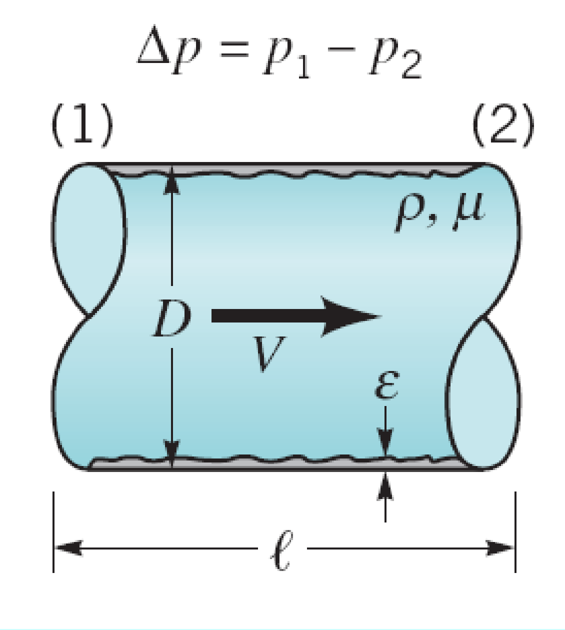
A similar witticism has been attributed to **Horace Lamb** (who had published a noted text book on Hydrodynamics)—his choice being quantum mechanics (instead of relativity) and turbulence. Lamb was quoted as saying in a speech to the British Association for the Advancement of Science in 1932, "I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is **quantum electrodynamics**, and the other is the **turbulent motion of fluids**. And about the former I am rather optimistic."

8.4 Dimensional Analysis of Pipe Flow

- Most turbulent pipe flow analyses are based on experimental data and semi-empirical formulas.

  $h_L = h_{Lmajor} + h_{Lminor}$

- Major losses : loss associated with the straight portion of the pipe $\Delta p = F\left(V, D, \ell, \varepsilon, \mu, \rho\right)$
- pressure drop and head loss in a pipe
  - For laminar flow the pressure drop is found to be independent of the roughness of the pipe.
  - For turbulent flow, the pressure drop is expected to be a function of the roughness.



Pressure drop

- For turbulent flow, the pressure drop is expected to be a function of the roughness.

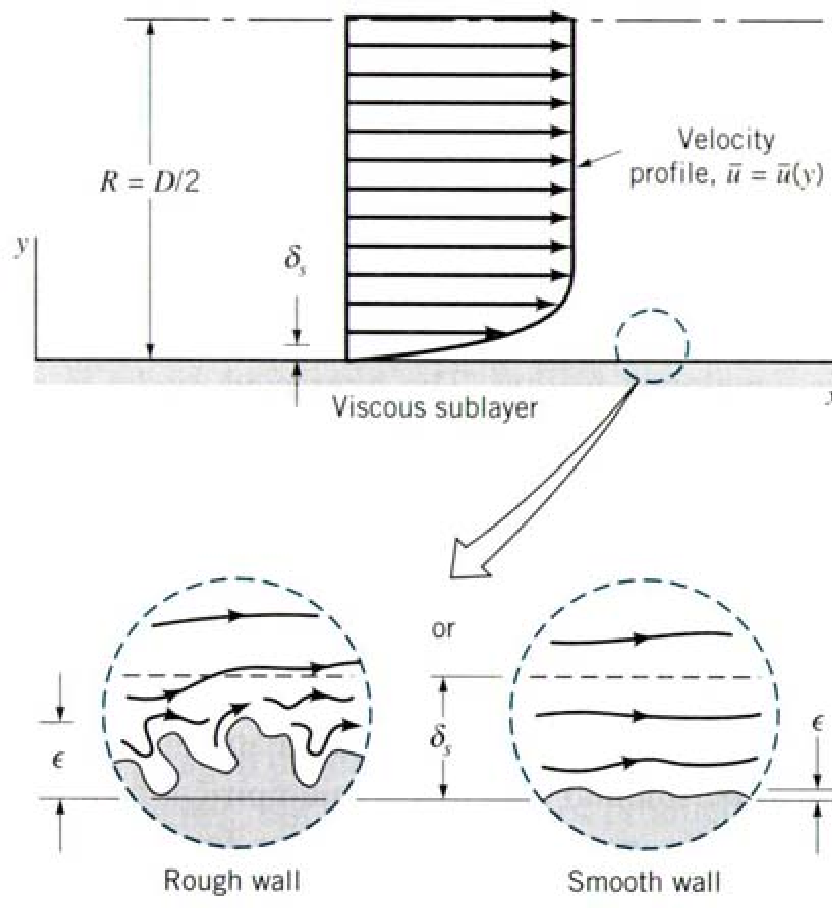


TABLE 8.1

Equivalent Roughness for New Pipes [From Moody (Ref. 7) and Colebrook (Ref. 8)]

	Equivalent Roughness, ε	
Pipe	**Feet**	**Millimeters**
Riveted steel	0.003–0.03	0.9–9.0
Concrete	0.001–0.01	0.3–3.0
Wood stave	0.0006–0.003	0.18–0.9
Cast iron	0.00085	0.26
Galvanized iron	0.0005	0.15
Commercial steel or wrought iron	0.00015	0.045
Drawn tubing	0.000005	0.0015
Plastic, glass	0.0 (smooth)	0.0 (smooth)

Pressure drop in pipe

- We will consider $0 \le \varepsilon / D \le 0.05$

  $k - r = 7 - 3 = 4$

  $$\frac{\Delta p}{\frac{1}{2}\rho V^2} = \tilde{\phi}\left(\frac{\rho V D}{\mu}, \frac{\ell}{D}, \frac{\varepsilon}{D}\right)$$

- Experiments indicate that pressure drop is proportional to the pipe length.

$$\frac{\Delta p}{\frac{1}{2}\rho V^2} = \frac{\ell}{D}\phi\left(Re, \frac{\varepsilon}{D}\right) \quad , \quad \Delta p = f\,\frac{\ell}{D}\frac{\rho V^2}{2}$$ - valid for horizontal pipe

or

$$f = \frac{\Delta p D}{\frac{1}{2}\rho V^2 \ell} = \phi\left(Re, \frac{\varepsilon}{D}\right)$$

$\uparrow$

friction factor

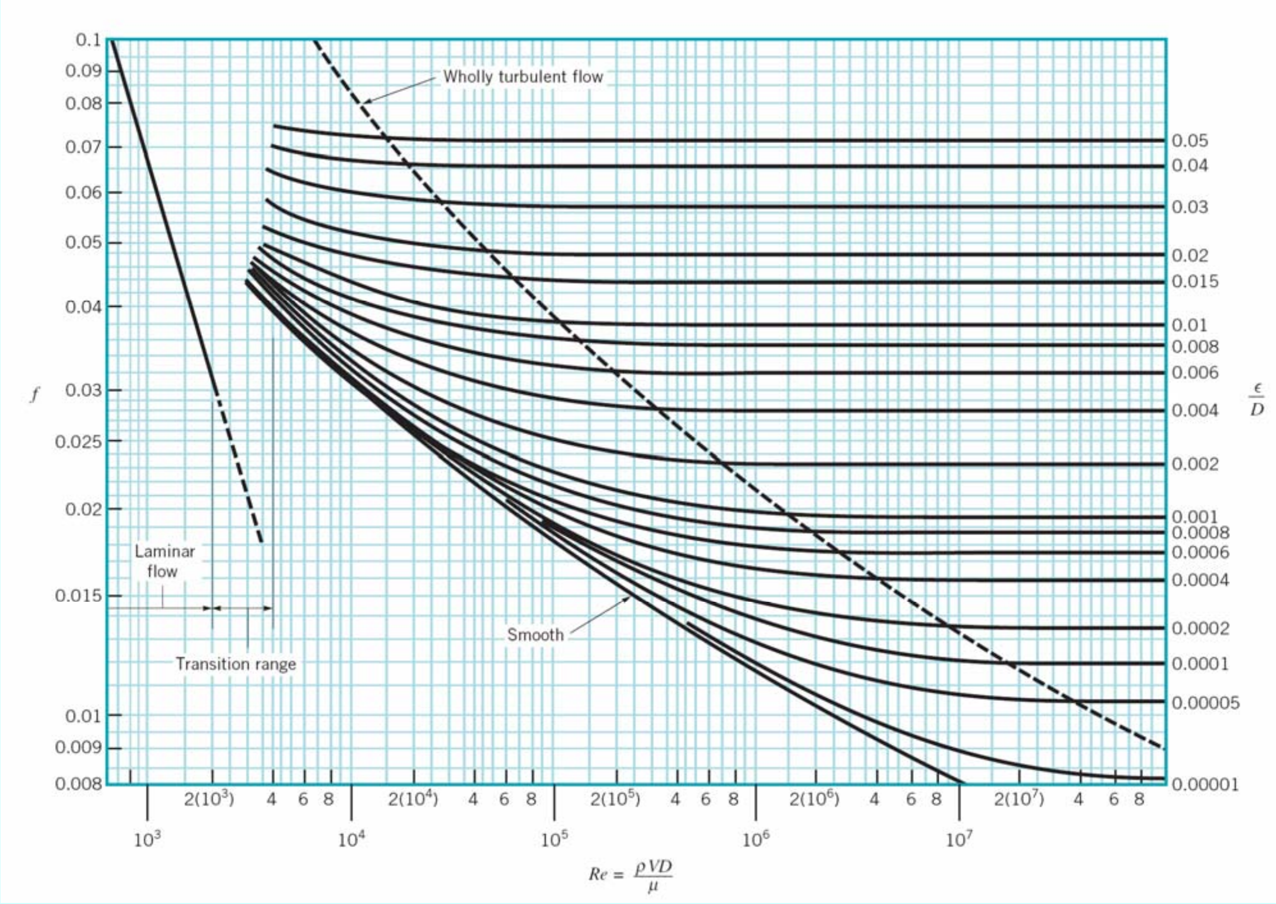
Friction coefficient

- For laminar flow

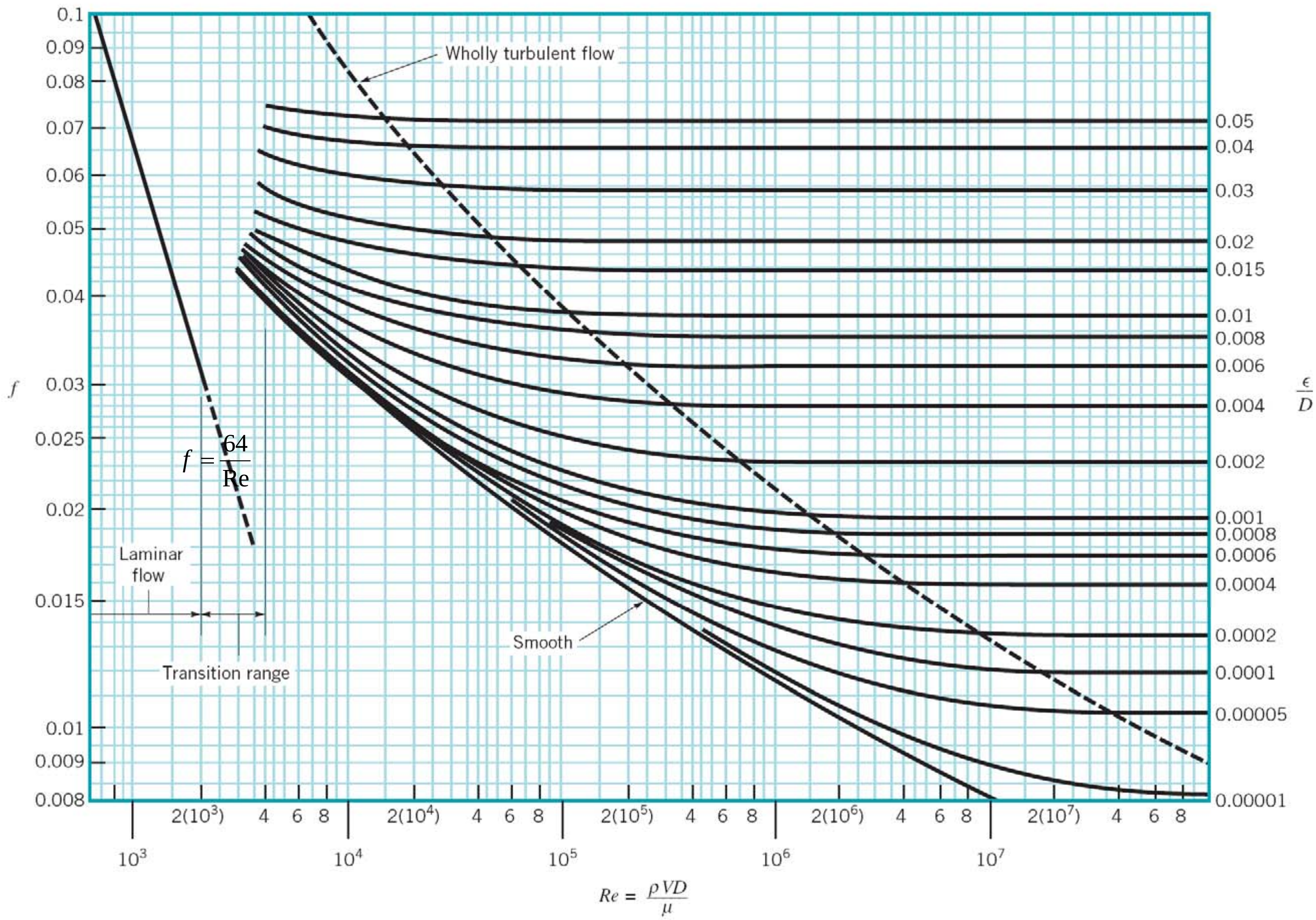
$$f = \frac{64}{\text{Re}} \quad \text{independent of} \quad \frac{\varepsilon}{D}$$

- For turbulent flow

$$f = \phi\left(Re, \frac{\varepsilon}{D}\right)$$



Moody chart



Wholly turbulent flow
0.1
0.09
0.08
0.07
0.06
0.05
0.04
f
0.03
0.025
0.02
0.015
0.01
0.009
0.008
$f = \frac{64}{\text{Re}}$
Laminar flow
Transition range
Smooth
0.05
0.04
0.03
0.02
0.015
0.01
0.008
0.006
0.004
0.002
0.001
0.0008
0.0006
0.0004
0.0002
0.0001
0.00005
0.00001
$\frac{\epsilon}{D}$
10^3 $2(10^3)$ 4 6 8 10^4 $2(10^4)$ 4 6 8 10^5 $2(10^5)$ 4 6 8 10^6 $2(10^6)$ 4 6 8 10^7 $2(10^7)$ 4 6 8
$Re = \frac{\rho VD}{\mu}$

- Energy Equation

$$\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_L$$ (cf. Eq. 5.89 in Section 5.3.4)

if $D_1 = D_2$, $z_1 = z_2$ (horizontal pipe) with fully developed flow $\alpha_1 = \alpha_2$

thus

$$\Delta p = p_1 - p_2 = \gamma h_L = f \frac{\ell}{D} \frac{\rho V^2}{2}$$

$$h_L = f \frac{\ell}{D} \frac{V^2}{2g}$$ - Darcy – Weisbach equation

valid for fully developed steady incompressible pipe flow horizontal or on a hill.

Determination of friction coefficient

- In general with

$V_1 = V_2$

$$p_1 - p_2 = \gamma\left(z_2 - z_1\right) + \gamma h_L = \gamma\left(z_2 - z_1\right) + f\,\frac{\ell}{D}\frac{\rho V^2}{2}$$

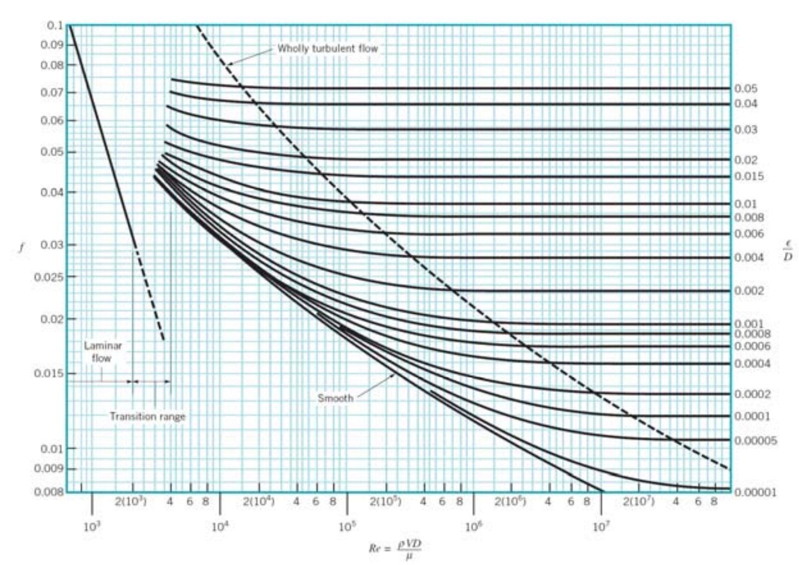
Part of the pressure change is due to the elevation change and part is due to the heat loss associated with friction effects.

- Moody chart

  10% accuracy is best expected

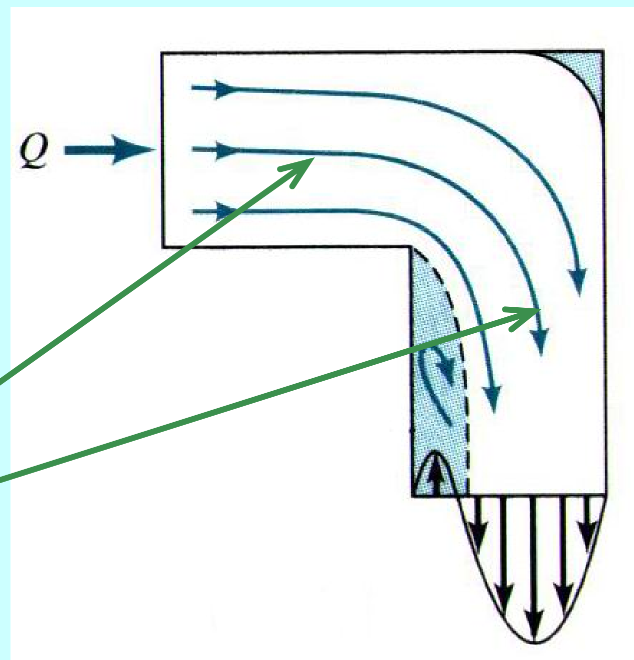
- Colebrook formula

$$\frac{1}{\sqrt{f}} = -2.0\,\log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right)$$

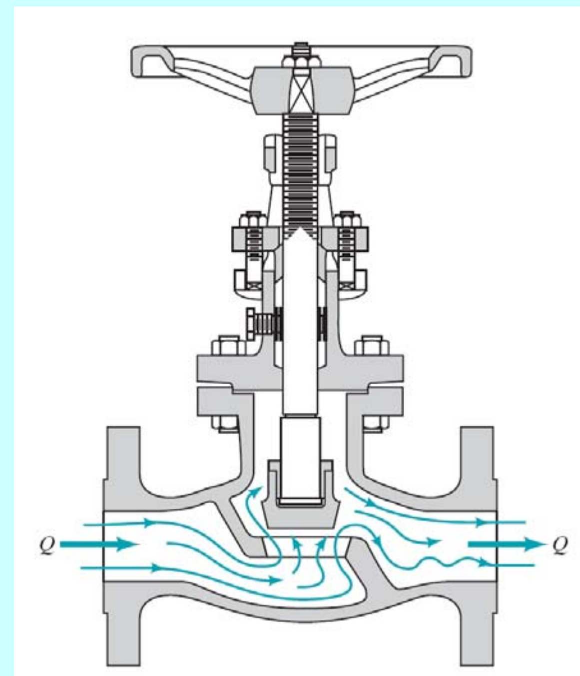


8.4.2 Minor Losses

- Major losses : loss associated with the straight portion of the pipe
- Minor losses : loss associated with valves, bends, tees,………..



Minor losses



Loss coefficients for minor loss

- The head loss information for essentially all components is given in dimensionless form and based on experimental data.
- loss coefficient

$$K_L = \frac{h_L}{\left(V^2/2g\right)} = \frac{\Delta p}{\frac{1}{2}\rho V^2}$$

so that $\Delta p = K_L \cdot \frac{1}{2}\rho V^2$

or

$$h_L = K_L \cdot \frac{V^2}{2g}$$

$$K_L = \phi\left(\text{geometry, Re}\right)$$

Loss coefficients for minor loss

For many practical applications, Re is large, the flow is dominated by inertia effect.

$$\Delta p \propto \tfrac{1}{2}\rho V^2$$

$$\text{thus } K_L = \phi(\text{geometry})$$

- Equivalent length

$$h_L = K_L \frac{V^2}{2g} = f \frac{\ell_{eq}}{D}\frac{V^2}{2g}$$

$$\text{or} \quad \ell_{eq} = \frac{K_L D}{f}$$

- loss coefficients
  An obvious way to reduce loss is to round the entrance region.

Entrance and Exit Flows

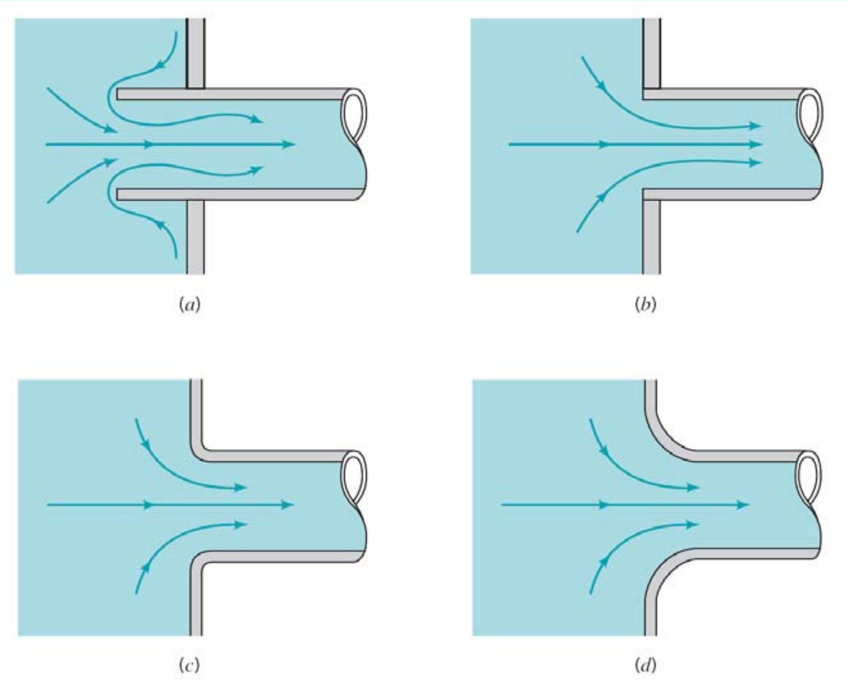


Figure 8.22 (p. 438)

Entrance flow conditions and loss coefficient (Refs. 28, 29). (*a*) Reentrant, K_L = 0.8, (*b*) sharp-edged, K_L = 0.5, (*c*) slightly rounded, K_L = 0.2 (see Fig. 8.24), (*d*) well-rounded, K_L = 0.04 (see Fig. 8.24).

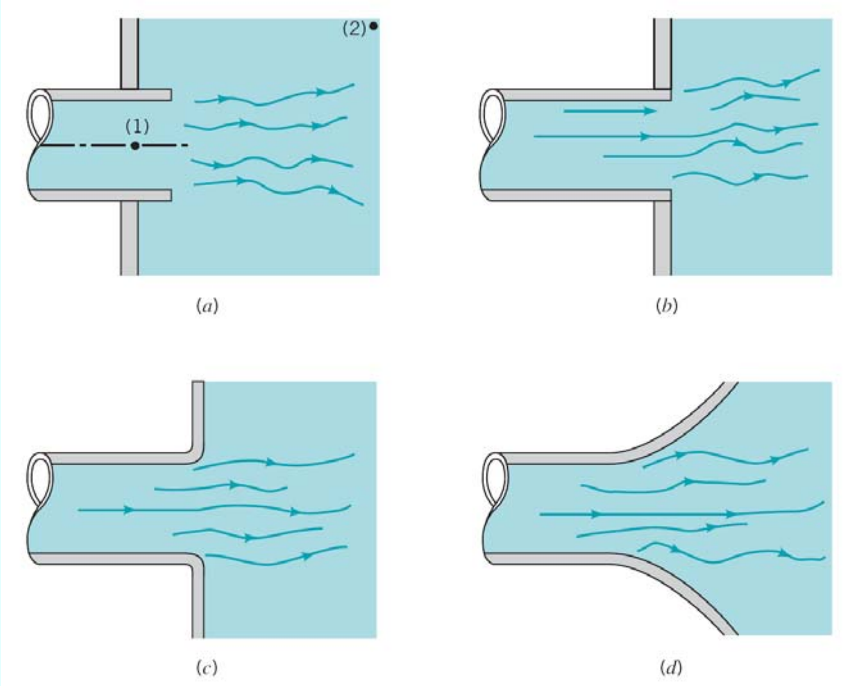


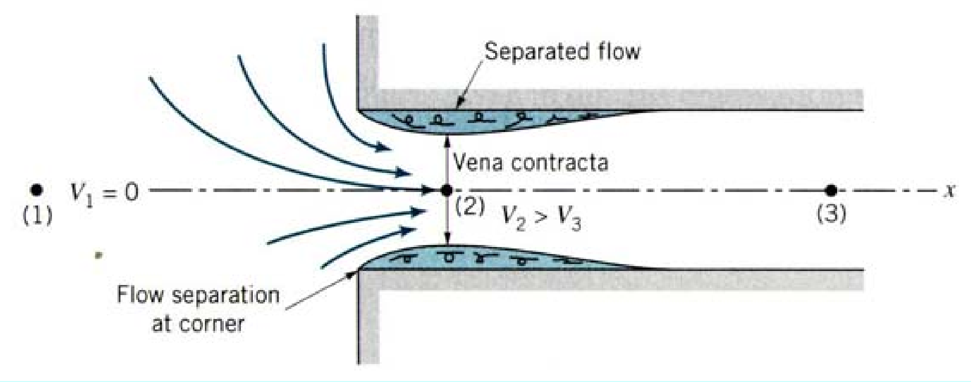
Figure 8.25 (p. 440)

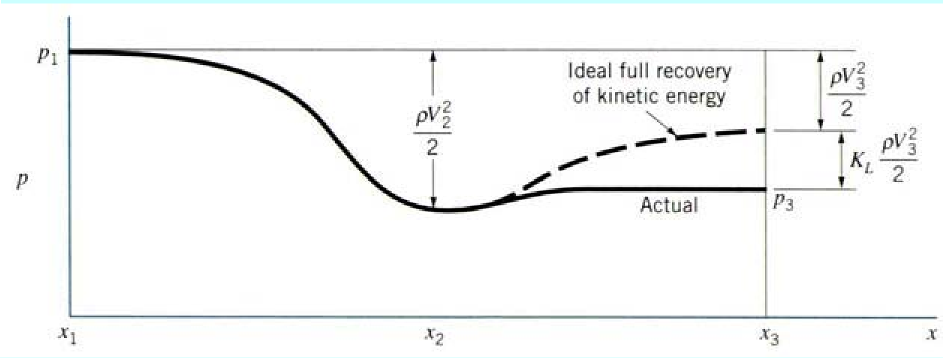
Exit flow conditions and loss coefficient. (*a*) Reentrant, K_L = 1.0, (*b*) sharp-edged, K_L = 1.0, (*c*) slightly rounded, K_L = 1.0, (*d*) well-rounded, K_L = 1.0.

V8.10 Entrance/exit flows

Pressure loss

- Flow patter and pressure distribution for a sharp edge entrance.

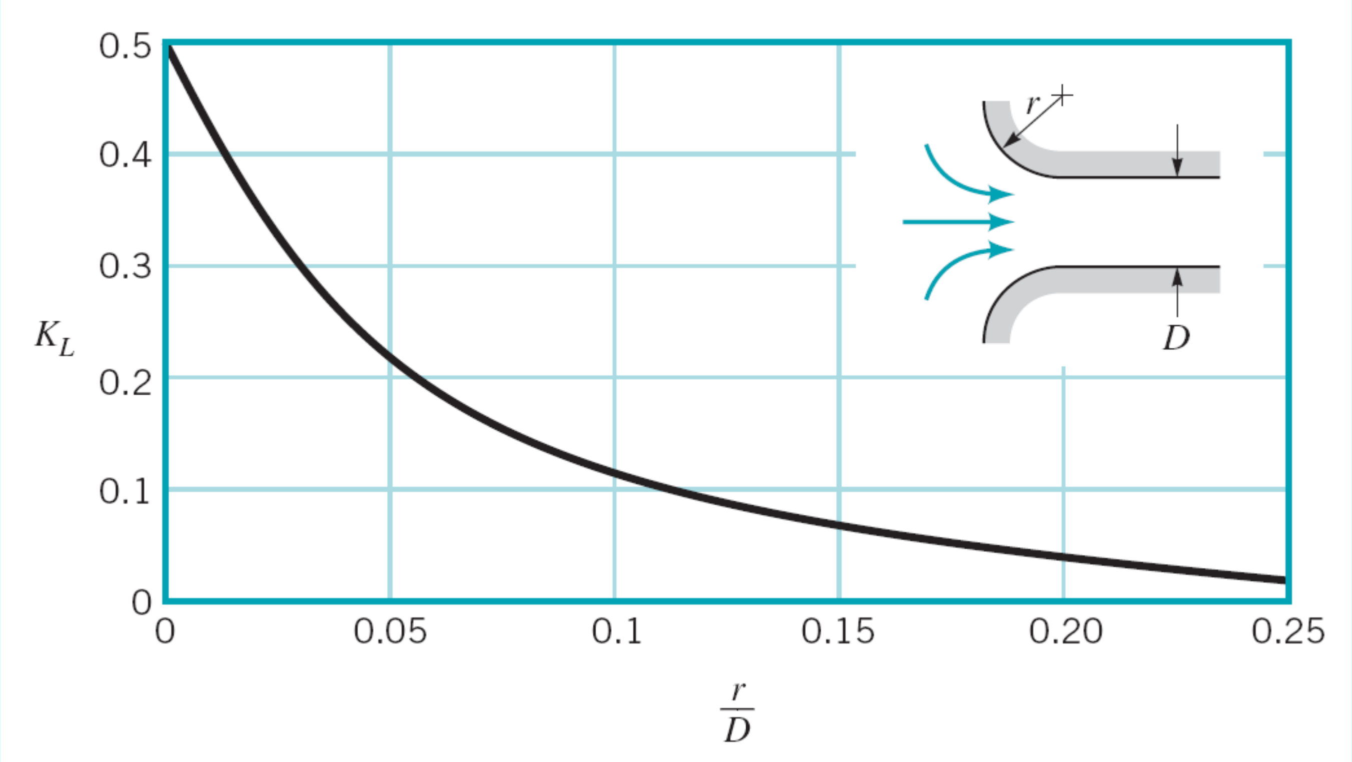




The majority of the loss is due to inertia effects that are eventually dissipated by the shear stresses within the fluid.
Only a small portion of the loss is due to the wall shear stress within the entrance region.

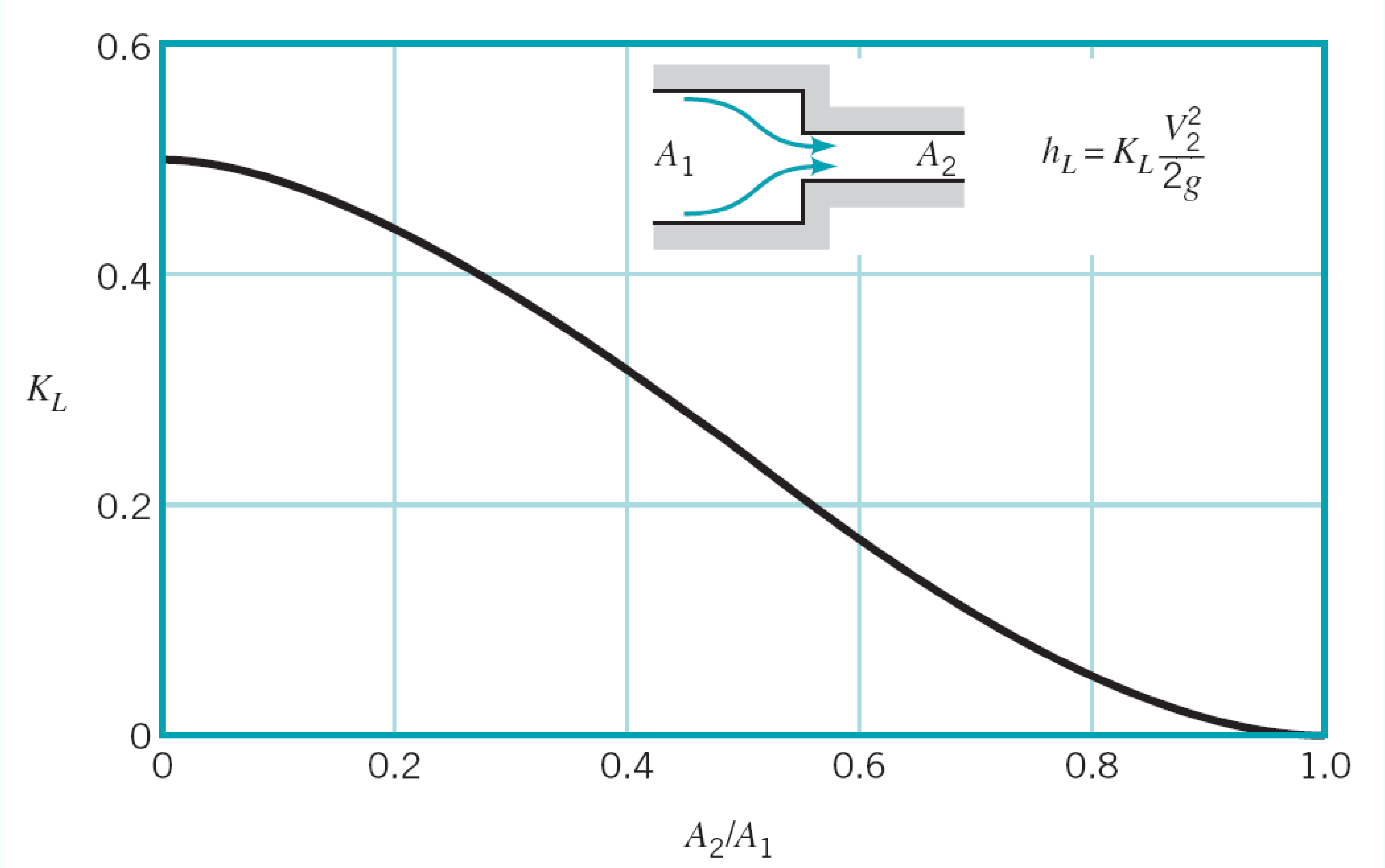
Loss coefficient

- Loss coefficients



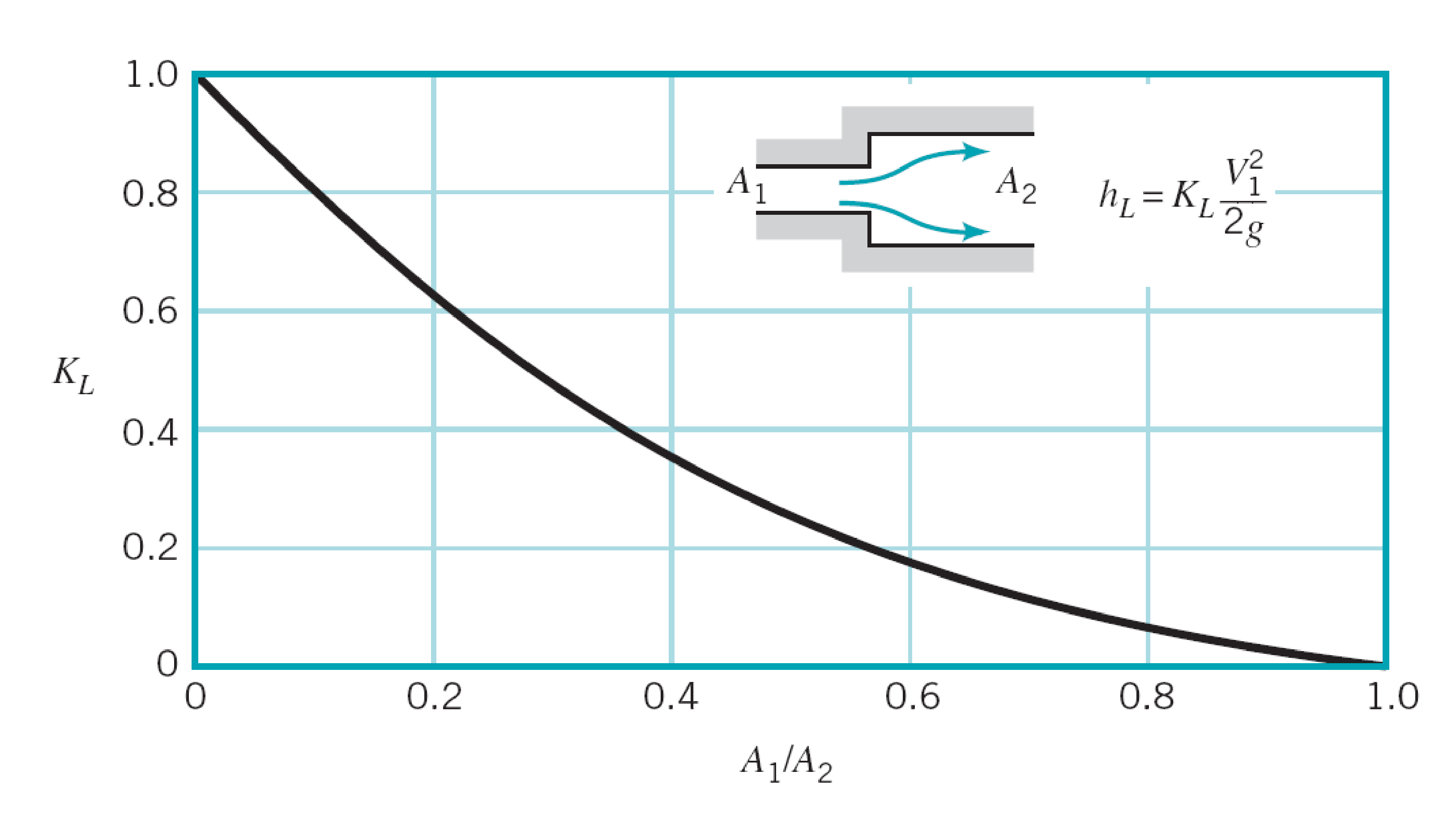
Loss coefficient

- Loss coefficients



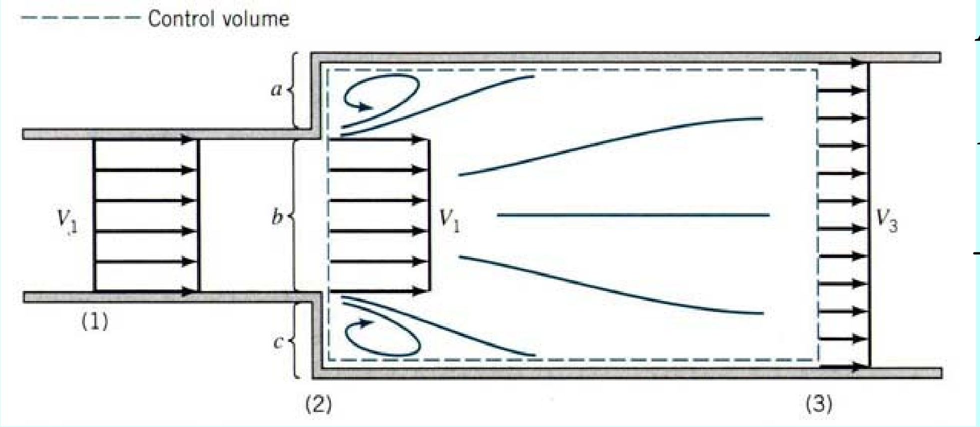
Loss coefficient

- Loss coefficients



Loss coefficient

- Loss coefficients for a sudden expansion



$$A_1V_1 = A_3V_3$$

$$p_1A_3 - p_3A_3 = \rho A_3V_3\left(V_3 - V_1\right)$$

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_3}{\gamma} + \frac{V_3^2}{2g} + h_L$$

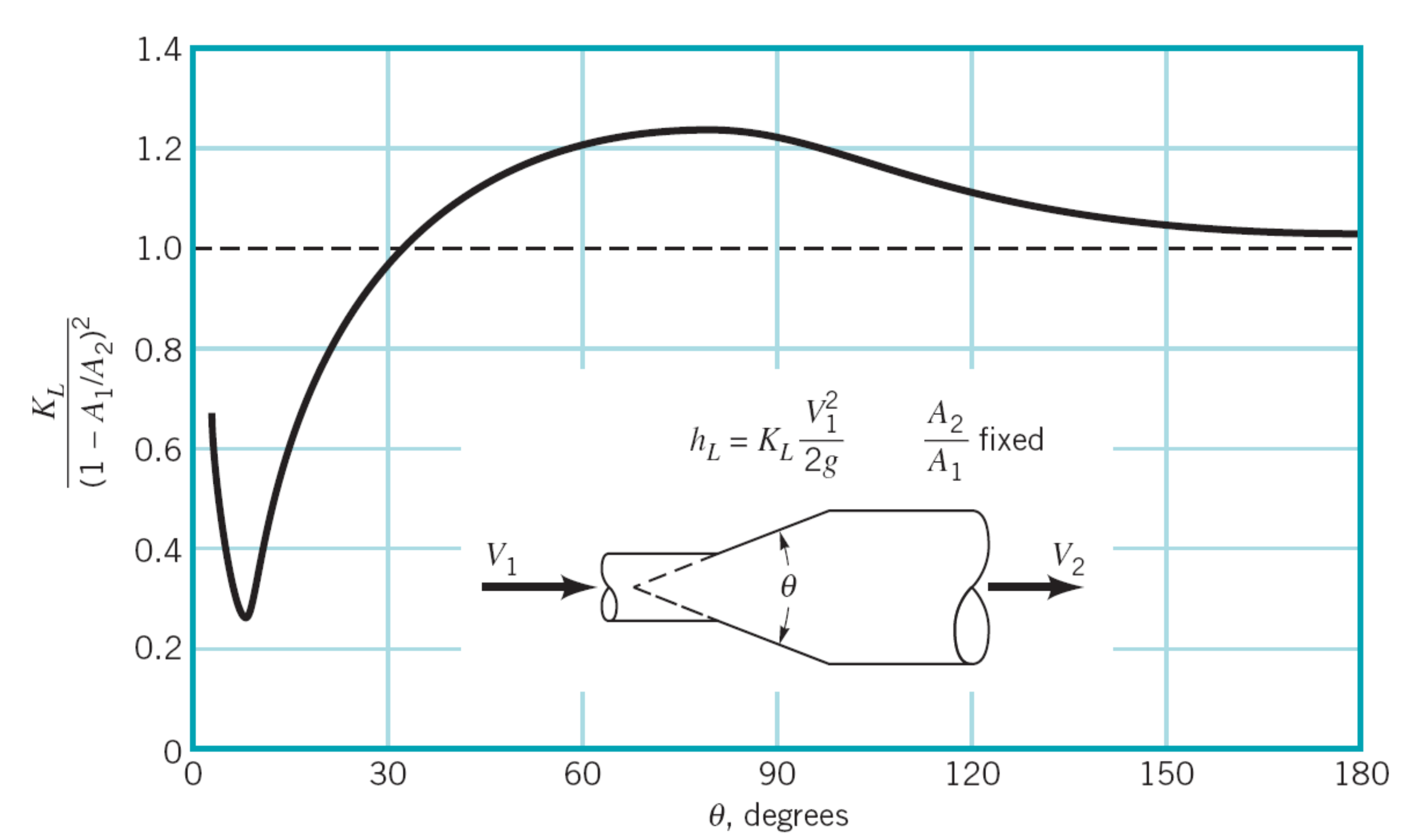
$$K_L = \frac{h_L}{V_1^2/2g} = \left(1 - \frac{A_1}{A_2}\right)^2$$

$$p_a = p_b = p_c = p_1$$

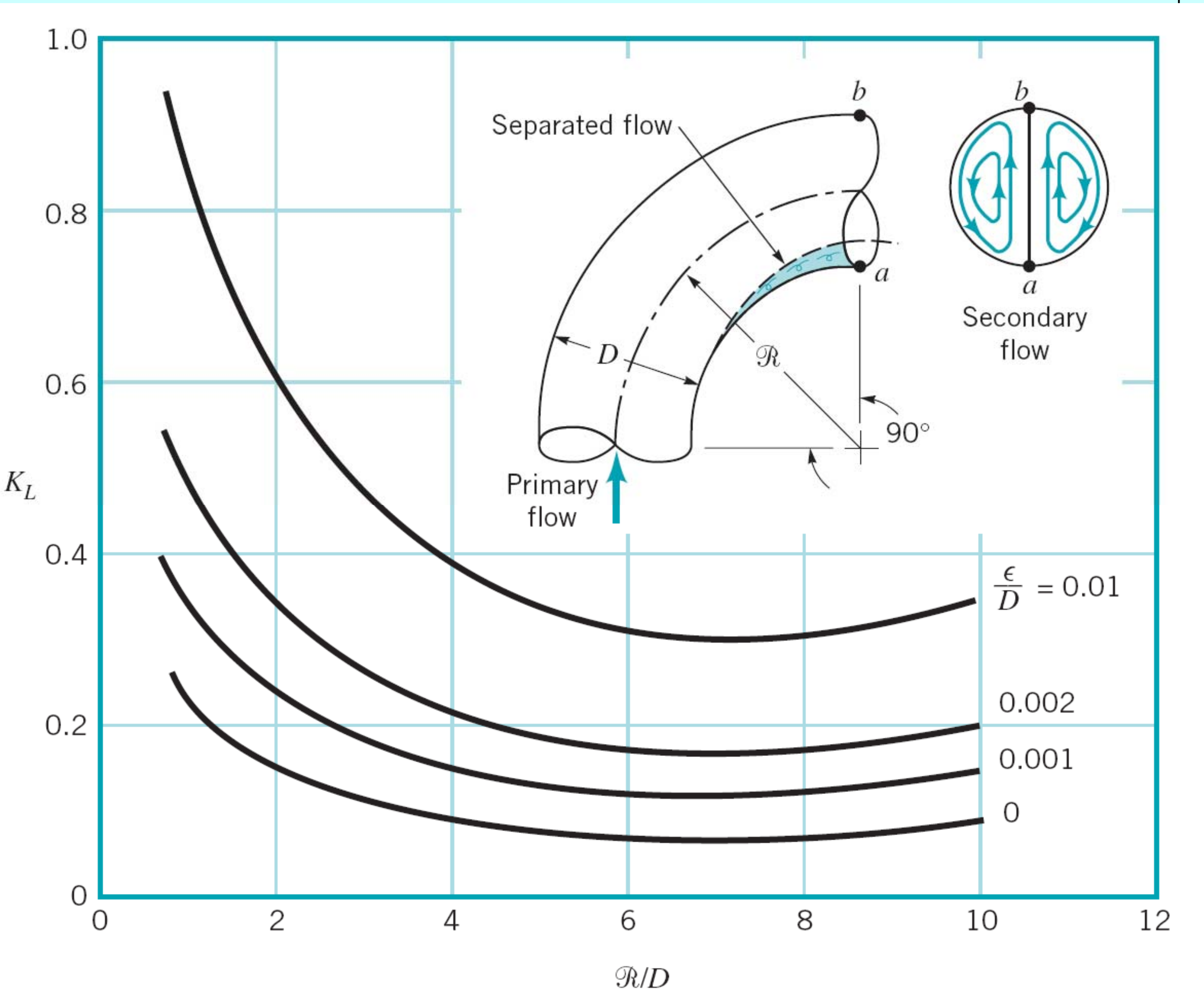
$$A_2 = A_3$$

V8.11 Separated flow in a diffuser

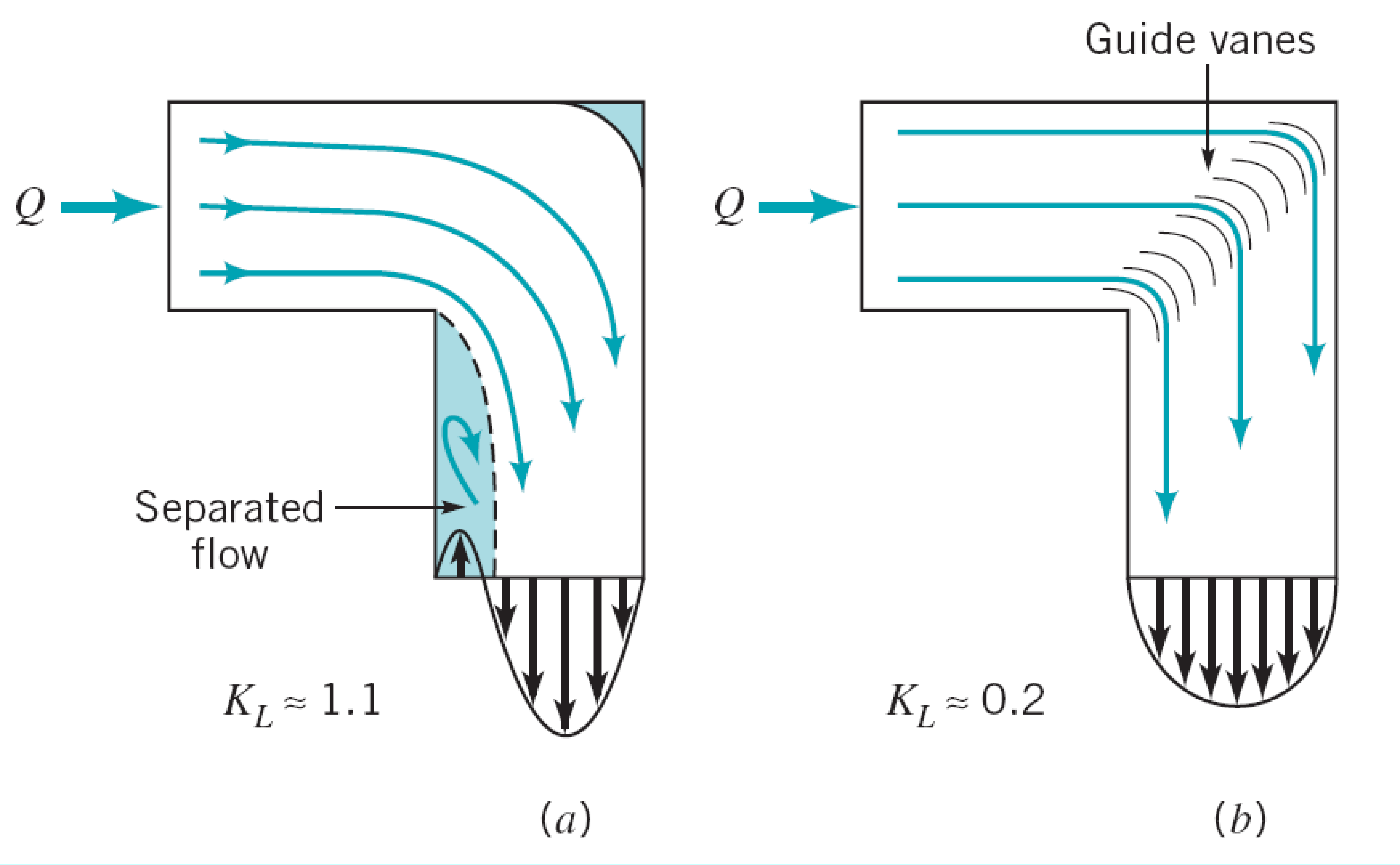
Loss coefficient



Loss coefficient



Loss coefficient

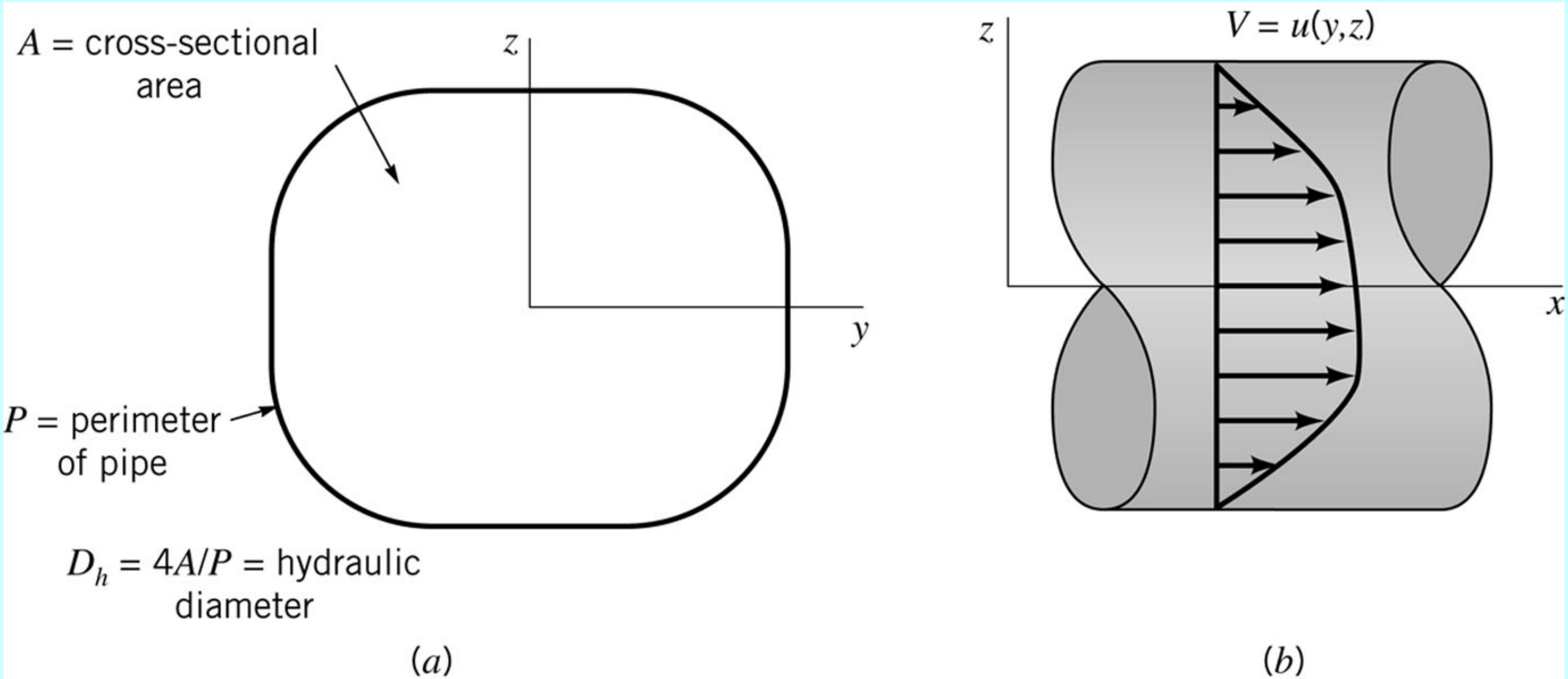


8.4.3 Noncircular Conduits

- For noncircular duct

$$f = \frac{C}{\mathrm{Re}_h} \quad \text{where } \mathrm{Re}_h = \frac{\rho V D_h}{\mu},\ D_h = \frac{4A}{P}$$

$$h_L = f\frac{\ell}{D_h}\frac{V^2}{2g}, \qquad \text{relative roughness, } \frac{\varepsilon}{D_h}$$

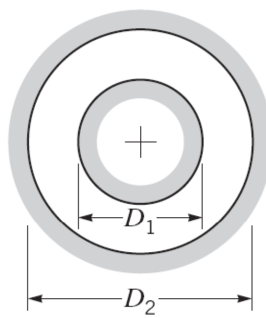
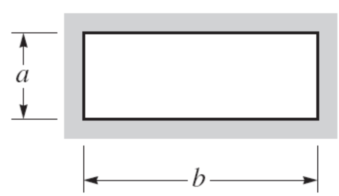


Noncircular Conduits

- For noncircular duct $f = \dfrac{C}{\text{Re}_h}, \; h_L = f\,\dfrac{\ell}{Dh}\,\dfrac{V^2}{2g}, \;$ relative roughness, $\dfrac{\varepsilon}{D_h}$
- C value

TABLE 8.3

Friction Factors for Laminar Flow in Noncircular Ducts (Data from Ref. 18)

Shape	Parameter	$C = f\,\text{Re}_h$
I. Concentric Annulus	D_1/D_2	
$D_h = D_2 - D_1$	0.0001	71.8
	0.01	80.1
	0.1	89.4
	0.6	95.6
	1.00	96.0
II. Rectangle	a/b	
$D_h = \dfrac{2ab}{a+b}$	0	96.0
	0.05	89.9
	0.10	84.7
	0.25	72.9
	0.50	62.2
	0.75	57.9
	1.00	56.9